Chapter 6
Applications of Integration
6.9 Calculus of the Hyperbolic Functions

Section Exercises

377. **[T]** Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

Answer: e^x and e^{-x}

378. From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

Answer: Answers may vary

379. Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Answer: Answers may vary

380. Use the quotient rule to verify that $\tanh(x)' = \operatorname{sech}^2(x)$.

Answer: Answers may vary

381. Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

Answer: Answers may vary

382. Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

Answer: $2\cosh(x)\sinh(x)$

383. Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

Answer: Answers may vary

384. Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

Answer: $\cosh(x)\cosh(y) + \sinh(y)\sinh(x)$

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

385. **[T]** $\cosh(3x+1)$

Answer: $3\sinh(3x+1)$

386. **[T]** $\sinh(x^2)$

Answer: $2x\cosh(x^2)$

387. **[T]** $\frac{1}{\cosh(x)}$

Answer: $-\tanh(x)\operatorname{sech}(x)$

388. **[T]** $\sinh(\ln(x))$

Answer: $\frac{1}{2}\left(\frac{1}{x^2}+1\right)$

389. **[T]** $\cosh^2(x)+\sinh^2(x)$

Answer: $4\cosh(x)\sinh(x)$

390. **[T]** $\cosh^2(x)-\sinh^2(x)$

Answer: 0

391. **[T]** $\tanh\left(\sqrt{x^2+1}\right)$

Answer: $\frac{x\operatorname{sech}^2\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

392. **[T]** $\frac{1+\tanh(x)}{1-\tanh(x)}$

Answer: $2(\sinh(2x)+\cosh(2x))$ $2e^{2x}$

393. **[T]** $\sinh^6(x)$

Answer: $6\sinh^5(x)\cosh(x)$

394. **[T** $\ln(\operatorname{sech}(x)+\tanh(x))$

Answer: $\frac{\operatorname{sech}(x)-\tanh(x)}{\sinh(x)+1}$

For the following exercises, find the antiderivatives for the given functions.

395. $\cosh(2x+1)$

Answer: $\frac{1}{2}\sinh(2x+1)+C$

396. $\tanh(3x+2)$

Answer: $\frac{1}{3}\ln(\cosh(3x+2))+C$

397. $x\cosh(x^2)$

Answer: $\frac{1}{2}\sinh^2(x^2)+C$

398. $3x^3\tanh(x^4)$

Answer: $\frac{3}{4}\ln(\cosh(x^4))+C$

399. $\cosh^2(x)\sinh(x)$

Answer: $\frac{1}{3}\cosh^3(x)+C$

400. $\tanh^2(x)\operatorname{sech}^2(x)$

Answer: $\frac{1}{3}\tanh^3(x)+C$

401. $\dfrac{\sinh(x)}{1+\cosh(x)}$

Answer: $\ln\left(1+\cosh(x)\right)+C$

402. $\coth(x)$

Answer: $\ln(\sinh(x))+C$

403. $\cosh(x)+\sinh(x)$

Answer: $\cosh(x)+\sinh(x)+C$

404. $(\cosh(x)+\sinh(x))^n$

Answer: $\dfrac{(\sinh(x)+\cosh(x))^n}{n}+C$

For the following exercises, find the derivatives for the functions.

405. $\tanh^{-1}(4x)$

Answer: $\dfrac{4}{1-16x^2}$

406. $\sinh^{-1}(x^2)$

Answer: $\dfrac{2x}{\sqrt{1+x^4}}$

407. $\sinh^{-1}(\cosh(x))$

Answer: $\dfrac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$

408. $\cosh^{-1}(x^3)$

Answer: $\dfrac{3x^2}{\sqrt{x^6-1}}$

409. $\tanh^{-1}(\cos(x))$

Answer: $-\csc(x)$

410. $e^{\sinh^{-1}(x)}$

Answer: $\dfrac{e^{\sin^{-1}(x)}}{\sqrt{x^2+1}}$

411. $\ln(\tanh^{-1}(x))$

Answer: $-\dfrac{1}{(x^2-1)\tanh^{-1}(x)}$

For the following exercises, find the antiderivatives for the functions.

412. $\displaystyle\int \frac{dx}{4-x^2}$

Answer: $\dfrac{1}{2}\tanh^{-1}\left(\dfrac{x}{2}\right)+C$

413. $\int \frac{dx}{a^2 - x^2}$

Answer: $\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$

414. $\int \frac{dx}{\sqrt{x^2+1}}$

Answer: $\sinh^{-1}(x)+C$

415. $\int \frac{xdx}{\sqrt{x^2+1}}$

Answer: $\sqrt{x^2+1}+C$

416. $\int -\frac{dx}{x\sqrt{1-x^2}}$

Answer: $\text{sech}^{-1}(x)+C$

417. $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

Answer: $\cosh^{-1}(e^x)+C$

418. $\int -\frac{2x}{x^4-1}$

Answer: $\tanh^{-1}(x^2)$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

419. Show that $v(t) = \sqrt{g}\tanh\left(\sqrt{gt}\right)$ satisfies this equation.

Answer: Answers may vary.

420. Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g - v^2} = dt$.

Answer: Answers may vary.

421. **[T]** Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

Answer: 37.30

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

422. Show that $S = \sinh(cx)$ satisfies this equation.

Answer: Answers may vary

423. Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Answer: $y = \frac{1}{c}\cosh(cx)$

424. Sketch the cable and determine how far down it sags at $x = 0$.

Answer: $\frac{1}{c}$

For the following exercises, solve each problem.

425. **[T]** A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

Answer: -0.521095

426. **[T]** A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

Answer: 9.4016 meters

427. **[T]** A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Answer: 10

428. A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

Answer: a

429. Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

Answer: This is a proof; therefore, no answer is provided.

430. Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

Answer: This is a proof; therefore, no answer is provided.

431. Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

Answer: This is a proof; therefore, no answer is provided.

432. Prove that $\left(\cosh(x) + \sinh(x)\right)^n = \cosh(nx) + \sinh(nx)$.

Answer: This is a proof; therefore, no answer is provided.

433. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

Answer: This is a proof; therefore, no answer is provided.

434. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

Answer: This is a proof; therefore, no answer is provided.

Chapter Review Exercises

***True or False*. Justify your answer with a proof or a counterexample.**

435. The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

Answer: False

436. If the force is constant, the amount of work to move an object from $x=a$ to $x=b$ is $F(b-a)$.

Answer: True

437. The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

Answer: False

438. If the half-life of seaborgium-266 is 360 ms, then $k=(\ln(2))/360$.

Answer: True

For the following exercises, use the requested method to determine the volume of the solid.

439. The volume that has a base of the ellipse $x^2/4+y^2/9=1$ and cross-sections of an equilateral triangle perpendicular to the $y-$axis. Use the method of slicing.

Answer: $32\sqrt{3}$

440. $y=x^2-x,$ from $x=1$ to $x=4,$ rotated around the*y*-axis using the washer method

Answer: $\frac{171\pi}{2}$

441. $x=y^2$ and $x=3y$ rotated around the *y*-axis using the washer method

Answer: $\frac{162p}{5}$

442. $x=2y^2-y^3,$ $x=0,$ and $y=0$ rotated around the *x*-axis using cylindrical shells

Answer: $\frac{16\pi}{5}$

For the following exercises, find

a. the area of the region,

b. the volume of the solid when rotated around the *x*-axis, and

c. the volume of the solid when rotated around the *y*-axis. Use whichever method seems most appropriate to you.

443. $y = x^3,\ x = 0,\ y = 0,$ and $x = 2$

Answer: a. 4, b. $\frac{128\pi}{7}$, c. $\frac{64\pi}{5}$

444. $y = x^2 - x$ and $x = 0$

Answer: a. $\frac{1}{6}$, b. $\frac{\pi}{30}$, c. $\frac{\pi}{6}$

445. **[T]** $y = \ln(x) + 2$ and $y = x$

Answer: a. 1.949, b. 21.952, c. 17.099

446. $y = x^2$ and $y = \sqrt{x}$

Answer: a. $\frac{1}{3}$, b. $\frac{3\pi}{10}$, c. $\frac{3\pi}{10}$

447. $y = 5 + x,\ \ y = x^2,\ \ x = 0,$ and $x = 1$

Answer: a. $\frac{31}{6}$, b. $\frac{452\pi}{15}$, c. $\frac{31\rho}{6}$

448. Below $x^2 + y^2 = 1$ and above $y = 1 - x$

Answer: a. $\frac{1}{4}(\pi - 2)$, b. $\frac{\pi}{3}$, c. $\frac{\rho}{3}$

449. Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

Answer: 245.282

450. Find the center of mass for $r = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Answer: $x = 0$

451. Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

Answer: Mass: $\frac{1}{2}$, center of mass: $\left(\frac{18}{35}, \frac{9}{11}\right)$

For the following exercises, find the requested arc lengths.

452. The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

Answer: $\sinh(2)$

453. The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

Answer: $\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

454. The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the *y*-axis.

Answer: Volume: $\frac{44p}{3}$, surface area: $(20 + 8\sqrt{2})p$

455. The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the *x*-axis.

Answer: Volume: $\frac{3\pi}{4}$, surface area: $\pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$

456. For this exercise, consider the Karun-3 dam in Iran. Its shape can be approximated as an inverted isosceles triangle spanning across the river, with height 205 m and width (across the top of the dam) 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m^3. Find the total force on the wall of the dam.

Answer: 18,000,000,000 N

457. You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45°F$ outside and the temperature of the body is $78\,°F$. You know the cooling constant is $k = 0.00824°F/min$. When did the victim die, assuming that a human's temperature is $98°F$?

Answer: 11:02 a.m.

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

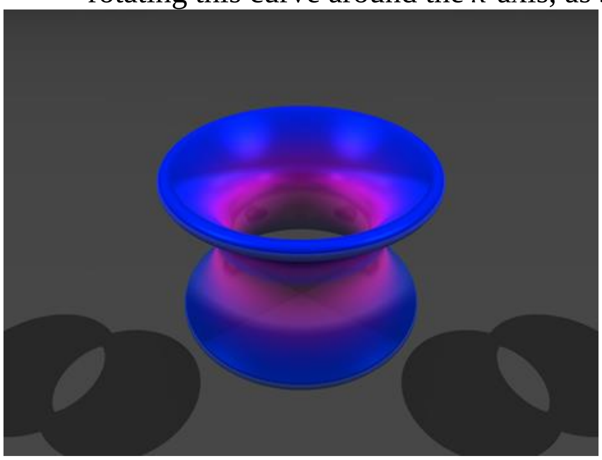
Years after 1920	**Value ($)**
1	63.90
3	100
5	110
7	160
9	381.17

458. **[T]** The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in 2014?

Answer: This model predicts that the rate of change of the Dow Jones is growing at an increasing rate, which is unsustainable. The 2014 prediction is 1.784×10^8.

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

459. Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the *x*-axis, as shown here.



Answer: $\pi(1 + \sinh(1)\cosh(1))$

460. Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the $x-$ axis.

Answer: $2\pi(1 + \sinh(1)\cosh(1))$