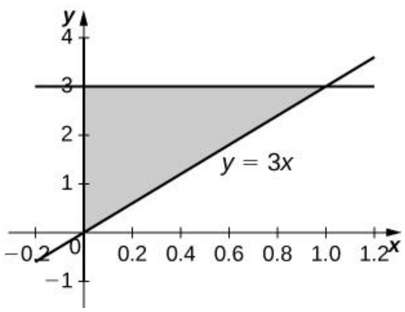
Chapter 6
Applications of Integration
6.3 Volumes of Revolution: Cylindrical Shells

Section Exercises

For the following exercise, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

114. **[T]** Bounded by the curves $y = 3x,$ $x = 0,$ and $y = 3$ rotated around the y-axis.

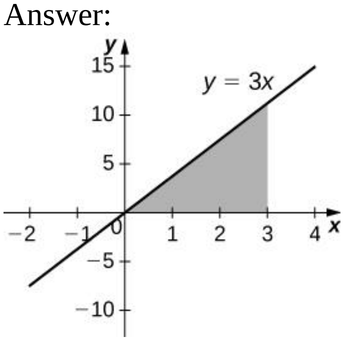
Answer:



π units^3

115. **[T]** Bounded by the curves $y = 3x,$ $x = 0,$ and $x = 3$ rotated around the y-axis.

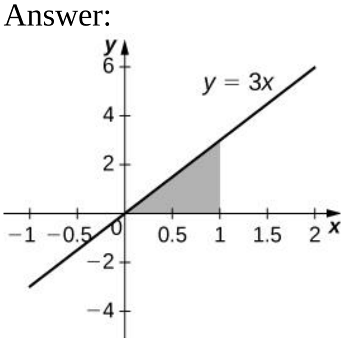
Answer:



54π units^3

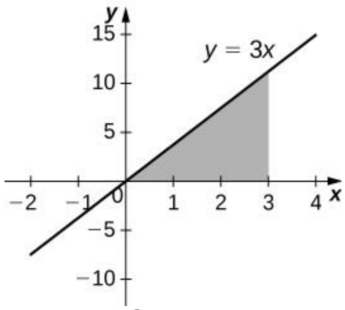
116. **[T]** Bounded by the curves $y = 3x,$ $x = 0,$ and $y = 3$ rotated around the x-axis.

Answer:



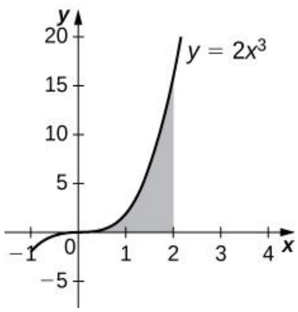
6π 54π units^3

117. **[T]** Bounded by the curves $y = 3x,\ x = 0,$ and $x = 3$ rotated around the x-axis.
Answer:



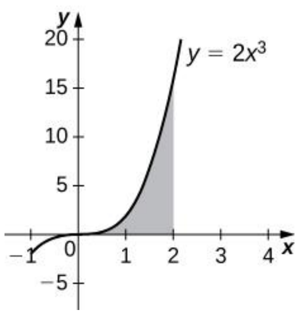
81π units^3

118. **[T]** Bounded by the curves $y = 2x^3,\ y = 0,$ and $x = 2$ rotated around the y-axis.
Answer:



$\frac{128\pi}{5}$ units^3

119. **[T]** Bounded by the curves $y = 2x^3,\ y = 0,$ and $x = 2$ rotated around the x-axis.
Answer:



$\frac{512\pi}{7}$ units^3

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the *x*-axis and are rotated around the *y*-axis.

120. $y = 1 - x^2,\ x = 0,$ and $x = 1$

Answer: $\frac{\pi}{2}$ units^3

121. $y = 5x^3,\ x = 0,\ \text{and } x = 1$

Answer: 2π units^3

122. $y = \frac{1}{x},\ x = 1,\ \text{and } x = 100$

Answer: 198π units^3

123. $y = \sqrt{1-x^2},\ x = 0,\ \text{and } x = 1$

Answer: $\frac{2\pi}{3}$ units^3

124. $y = \frac{1}{1+x^2},\ x = 0,\ \text{and } x = 3$

Answer: $\pi \ln(10)$ units^3

125. $y = \sin x^2,\ x = 0,\ \text{and } x = \sqrt{\pi}$

Answer: 2π units^3

126. $y = \frac{1}{\sqrt{1-x^2}},\ x = 0,\ \text{and } x = \frac{1}{2}$

Answer: $\pi\left(2-\sqrt{3}\right)$ units^3

127. $y = \sqrt{x},\ x = 0,\ \text{and } x = 1$

Answer: $\frac{4\pi}{5}$ units^3

128. $y = \left(1+x^2\right)^3,\ x = 0,\ \text{and } x = 1$

Answer: $\frac{15\pi}{4}$ units^3

129. $y = 5x^3 - 2x^4,\ x = 0,\ \text{and } x = 2$

Answer: $\frac{64\pi}{3}$ units^3

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the *x*-axis.

130. $y = \sqrt{1-x^2},\ x = 0,\ \text{and } x = 1$

Answer: $\frac{2\pi}{3}$ units^3

131. $y = x^2,\ x = 0,$ and $x = 2$

Answer: $\frac{32\pi}{5}$ units^3

132. $y = e^x,\ x = 0,$ and $x = 1$

Answer: $\frac{\pi}{2}\left(e^2 - 1\right)$ units^3

133. $y = \ln(x),\ x = 1,$ and $x = e$

Answer: $\pi(e-2)$ units^3

134. $x = \frac{1}{1+y^2},\ y = 1,$ and $y = 4$

Answer: $\pi \ln\left(\frac{17}{2}\right)$ units^3

135. $x = \frac{1+y^2}{y},\ y = 0,$ and $y = 2$

Answer: $\frac{28\pi}{3}$ units^3

136. $x = \cos y,\ y = 0,$ and $y = \pi$

Answer: -4π units^3

137. $x = y^3 - 4y^2,\ y = -1,$ and $y = 2$

Answer: $\frac{-84\pi}{5}$ units^3

138. $x = ye^y,\ y = 0,$ and $y = 2$

Answer: $2\pi\left(2e^2 - 2\right)$ units^3

139. $x = \cos ye^y,\ x = 0,$ and $x = \pi$

Answer: $-e^{\pi}\pi^2$ units^3

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

140. $y = 3 - x,\ y = 0,\ x = 0,$ and $x = 2$ rotated around the y-axis.

Answer: $\frac{20\pi}{3}$ units^3

141. $y = x^3,\ y = 0,$ and $y = 8$ rotated around the y-axis.

Answer: $\frac{64\pi}{5}$ units^3

142. $y = x^2,\ y = x,$ rotated around the y-axis.

Answer: $\frac{\pi}{6}$ units^3

143. $y = \sqrt{x},\ y = 0,$ and $x = 1$ rotated around the line $x = 2$.

Answer: $\frac{28\pi}{15}$ units^3

144. $y = \frac{1}{4 - x},\ x = 1,$ and $x = 2$ rotated around the line $x = 4$.

Answer: 2π units^3

145. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

Answer: $\frac{3\pi}{10}$ units^3

146. $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

Answer: $\frac{31\pi}{30}$ units^3

147. $x = y^3,\ x = \frac{1}{y},\ x = 1,$ and $x = 2$ rotated around the x-axis.

Answer: $\frac{52\pi}{5}$ units^3

148. $x = y^2$ and $y = x$ rotated around the line $y = 2$.

Answer: $\frac{\pi}{2}$ units^3

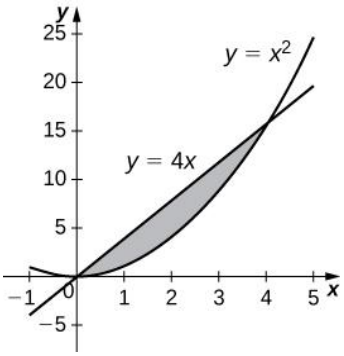
149. **[T]** Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

Answer: 0.9876 units^3

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

150. **[T]** $y = x^2$ and $y = 4x$ rotated around the y-axis.

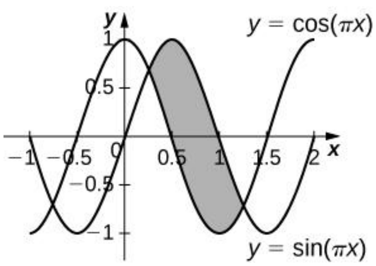
Answer:



$\frac{128\pi}{3}$ units^3

151. **[T]** $y = \cos(\pi x)$, $y = \sin(\pi x)$, $x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

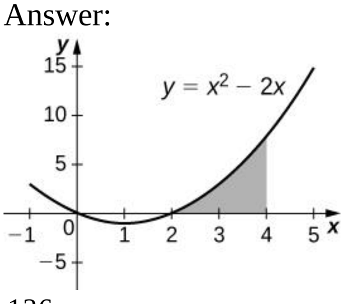
Answer:



$3\sqrt{2}$ units^3

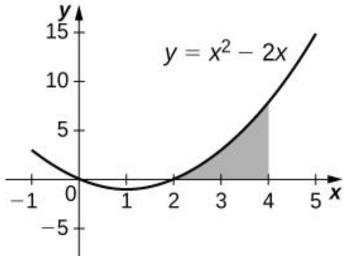
152. **[T]** $y = x^2 - 2x$, $x = 2$, and $x = 4$ rotated around the y-axis.

Answer:



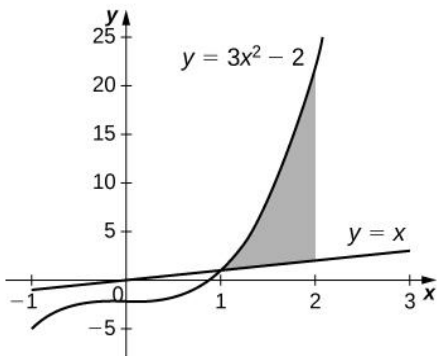
$\frac{136\pi}{3}$ units^3

153. **[T]** $y = x^2 - 2x,\ x = 2,$ and $x = 4$ rotated around the x-axis.
Answer:



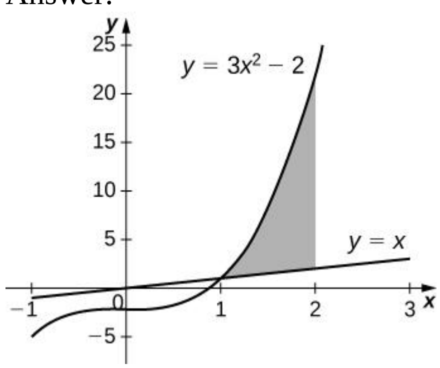
$\frac{496\pi}{15}$ units^3

154. **[T]** $y = 3x^3 - 2,\ y = x,$ and $x = 2$ rotated around the x-axis.
Answer:



$\frac{2519\pi}{21}$ units^3

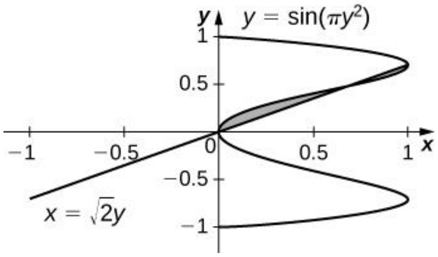
155. **[T]** $y = 3x^3 - 2,\ y = x,$ and $x = 2$ rotated around the y-axis.
Answer:



$\frac{398\pi}{15}$ units^3

156. **[T]** $x=\sin\left(\pi y^2\right)$ and $x=\sqrt{2}y$ rotated around the x-axis.

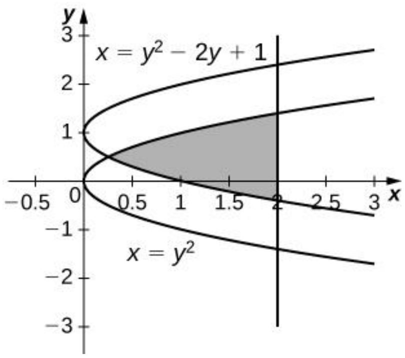
Answer:



$\frac{\pi}{3}-1$ units^3

157. **[T]** $x=y^2$, $x=y^2-2y+1$, and $x=2$ rotated around the y-axis.

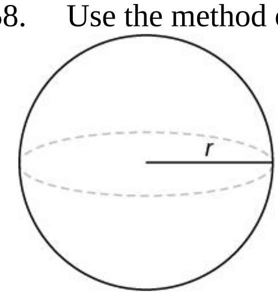
Answer:



15.9074 units^3

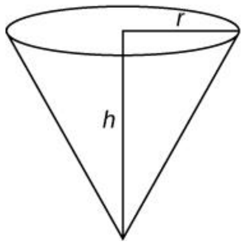
For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

158. Use the method of shells to find the volume of a sphere of radius r.



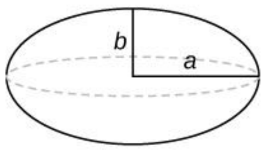
Answer: $\frac{4}{3}\pi r^3$ units^3

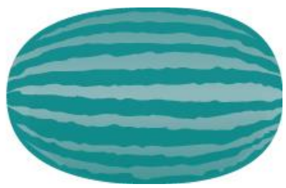
159. Use the method of shells to find the volume of a cone with radius r and height h.



Answer: $\frac{1}{3}\pi r^2 h$ units^3

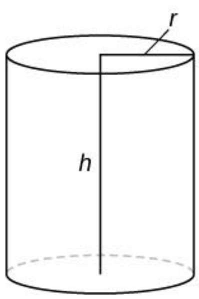
160. Use the method of shells to find the volume of an ellipse $\left(x^2/a^2\right)+\left(y^2/b^2\right)=1$ rotated around the x-axis.





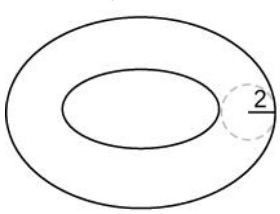
Answer: $\frac{4}{3}\pi ab^2$

161. Use the method of shells to find the volume of a cylinder with radius r and height h .



Answer: $\pi r^2 h$ units^3

162. Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



Answer: $6\pi^2$ units^3

163. Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a]$.

Answer: πa^2 units^3

164. Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint:* Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

Answer: $\int_0^1 2\pi x f(x)\,dx = \int_0^b \pi\left(f^{-1}(Y)^2\right)dy$ units^3