Chapter 6
Applications of Integration
6.6 Moments and Centers of Mass

Section Exercises

For the following exercises, calculate the center of mass for the collection of masses given.

254. $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

Answer: $\frac{5}{3}$

255. $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Answer: $\frac{5}{4}$

256. $m = 3$ at $x = 0, 1, 2, 6$

Answer: $\frac{9}{4}$

257. Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Answer: $\left(\frac{2}{3}, \frac{2}{3}\right)$

258. $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

Answer: $\left(\frac{1}{5}, \frac{4}{5}\right)$

259. $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Answer: $\left(\frac{7}{4}, \frac{3}{2}\right)$

For the following exercises, compute the center of mass $\overline{x}$.

260. $\rho = 1$ for $x \in (-1, 3)$

Answer: 1

261. $\rho = x^2$ for $x \in (0, L)$

Answer: $\frac{3L}{4}$

262. $\rho = 1$ for $x \in (0,1)$ and $\rho = 2$ for $x \in (1,2)$

Answer: $\frac{7}{6}$

263. $r = \sin x$ for $x \in (0,\pi)$

Answer: $\frac{\pi}{2}$

264. $r = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

Answer: $\frac{\pi}{2} - 1$

265. $\rho = e^x$ for $x \in (0,2)$

Answer: $\frac{e^2+1}{e^2-1}$

266. $\rho = x^3 + xe^{-x}$ for $x \in (0,1)$

Answer: $\frac{44e-100}{25e-40}$

267. $r = x\sin x$ for $x \in (0,\pi)$

Answer: $\frac{\pi^2-4}{\pi}$

268. $\rho = \sqrt{x}$ for $x \in (1,4)$

Answer: $\frac{93}{35}$

269. $\rho = \ln x$ for $x \in (1,e)$

Answer: $\frac{1}{4}\left(1+e^2\right)$

For the following exercises, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

270. $\rho = 7$ in the square $0 \le x \le 1, 0 \le y \le 1$

Answer: $\left(\frac{1}{2}, \frac{1}{2}\right)$

271. $\rho = 3$ in the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$

Answer: $\left(\frac{a}{3}, \frac{b}{3}\right)$

272. $\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x)$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

Answer: $(0,0)$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

273. **[T]** The region bounded by $y = \cos(2x)$, $x = -\frac{\rho}{4}$, and $x = \frac{\rho}{4}$

Answer: $\left(0, \frac{\pi}{8}\right)$

274. **[T]** The region between $y = 2x^2$, $y = 0$, $x = 0$, and $x = 1$

Answer: $\left(\frac{3}{4}, \frac{3}{5}\right)$

275. **[T]** The region between $y = \frac{5}{4}x^2$ and $y = 5$

Answer: $(0,3)$

276. **[T]** Region between $y = \sqrt{x}$, $y = \ln(x)$, $x = 1$, and $x = 4$

Answer: $(2.38495, 1.5558)$

277. **[T]** The region bounded by $y = 0$, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Answer: $\left(0, \frac{4}{\pi}\right)$

278. **[T]** The region bounded by $y = 0$, $x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Answer: $\left(\frac{8}{3\pi}, \frac{4}{\pi}\right)$

279. **[T]** The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

Answer: $\left(\frac{5}{8}, \frac{1}{3}\right)$

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

280. Rotating $y = mx$ around the x -axis between $x = 0$ and $x = 1$

Answer: $\frac{m^2\pi}{3}$

281. Rotating $y = mx$ around the y -axis between $x = 0$ and $x = 1$

Answer: $\frac{m\pi}{3}$

282. A general cone created by rotating a triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$ around the y -axis. Does your answer agree with the volume of a cone?

Answer: $\frac{\pi a^2 b}{3}$

283. A general cylinder created by rotating a rectangle with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) around the y -axis. Does your answer agree with the volume of a cylinder?

Answer: $\pi a^2 b$

284. A sphere created by rotating a semicircle with radius a around the y -axis. Does your answer agree with the volume of a sphere?

Answer: $\frac{4\pi a^3}{3}$

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

285. **[T]** Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

Answer: $\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

286. **[T]** Triangle: $y = x$, $y = 2 - x$, and $y = 0$

Answer: $\left(1, \frac{1}{3}\right)$

287. **[T]** Lens: $y = x^2$ and $y = x$

Answer: $\left(\frac{1}{2}, \frac{2}{5}\right)$

288. **[T]** Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

Answer: $(0, 0)$

289. **[T]** Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

Answer: $\left(0, \frac{28}{9\pi}\right)$

290. Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

Answer: Center of mass: $\left(\frac{(a+1)(b+1)}{(a+2)(b+2)}, \frac{(a+1)(b+1)}{(2a+1)(2b+1)}\right)$, volume: $\frac{2\pi(a-b)}{(a+2)(b+2)}$

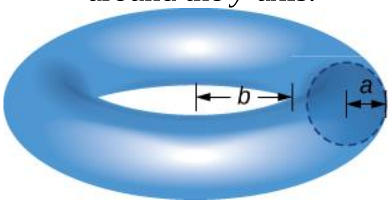
291. Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

Answer: Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$, volume: $\frac{2\pi a^4}{9}$

292. Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

Answer: Center of mass: $\left(\frac{\pi}{2a}, \frac{\pi b}{8}\right)$, volume: $\frac{2\pi^2 b}{a^2}$

293. Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the *y*-axis.



Answer: Volume: $2\pi^2 a^2 (b + a)$

294. Find the center of mass $(\overline{x}, \overline{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint:* Use the theorem of Pappus.)

Answer: $\dfrac{2}{\pi}$

Student Project
The Grand Canyon Skywalk

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.

Answer: Let A_1 denote the area of R_1. Then

$$A_1 = \frac{\pi}{2}\left[35^2 - 25^2\right] = \frac{\pi}{2}(600) = 300\pi \approx 942\ \text{ft}^2.$$

Let A_2 denote the area of R_2. Then

$$A_2 = 2\cdot 10\cdot 35 = 700\ \text{ft}^2.$$

Let A_3 denote the area of R_3. Then

$$A_3 = 2\cdot 10\cdot 48 = 960\ \text{ft}^2.$$

2. Determine the mass associated with each of the three sub-regions.

Answer: The total platform weight is 1.2 million pounds. Dividing by *g*, we see that the mass of the platform m_p is

$$m_p = \frac{1,200,000}{32} = 37,500\ \text{slugs}$$

Then the density of the lamina is

$$\rho = \frac{37,500}{A_1 + A_2 + A_3} = \frac{37,500}{2,602}$$

If m_1, m_2, and m_3 denote the masses associated with each of the three sub-regions, we have

$$m_1 = \rho A_1 = 942\rho = \frac{37,500}{2,602}(942) \approx 13,576.1\ \text{slugs}$$

$$m_2 = \rho A_2 = 700\rho = \frac{37,500}{2,602}(700) \approx 10.088.4\ \text{slugs}$$

$$m_3 = \rho A_3 = 960\rho = \frac{37,500}{2,602}(960) \approx 13,835.5\ \text{slugs}$$

3. Calculate the center of mass of each of the three sub-regions.

Answer: Let $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_2, \bar{y}_2)$, and $(\bar{x}_3, \bar{y}_3)$ denote the centers of mass of the three regions. We note that all three regions are symmetric with respect to the *y*-axis. Thus

$$\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$$

Additionally, R_2 and R_3 are symmetric vertically, so

$$\bar{y}_2 = -17.5$$

$$\bar{y}_3 = -59$$

To find $\bar{y}_1$, we need to find the moment with respect to the *x*-axis, M_{x1}. Looking at R_1, we see it is bounded above by

$$f(x) = \sqrt{35^2 - x^2},$$

and below by

$$g(x) = \begin{cases} 0 & -35 \le x \le -25 \\ \sqrt{25^2 - x^2} & -25 \le x \le 25 \\ 0 & 25 \le x \le 35 \end{cases}$$

Then

$$\begin{aligned} M_{x1} &= \frac{\rho}{2}\int_{-35}^{35}\left[\left(f(x)\right)^2 - \left(g(x)\right)^2\right] dx \\ &= \frac{\rho}{2}\int_{-35}^{-25}\left[\left(f(x)\right)^2 - (0)^2\right] dx + \int_{-25}^{25}\left[\left(f(x)\right)^2 - \left(g(x)\right)^2\right] dx + \int_{25}^{35}\left[\left(f(x)\right)^2 - (0)^2\right] dx \\ &= \frac{\rho}{2}\int_{-35}^{-25}\left(f(x)\right)^2 dx + \int_{-25}^{25}\left(f(x)\right)^2 dx - \int_{-25}^{25}\left(g(x)\right)^2 dx + \int_{25}^{35}\left(f(x)\right)^2 dx \\ &= \frac{\rho}{2}\int_{-35}^{35}\left(f(x)\right)^2 dx - \int_{-25}^{25}\left(g(x)\right)^2 dx \\ &= \frac{\rho}{2}\int_{-35}^{35}\left(35^2 - x^2\right) dx - \int_{-25}^{25}\left(25^2 - x^2\right) dx \\ &= \rho\left(\frac{54,500}{3}\right) \end{aligned}$$

Therefore we have

$$\bar{y}_1 = \frac{M_{x1}}{m_1} = \frac{\rho\left(\frac{54,500}{3}\right)}{942\rho} \approx 19.3$$

In summary, we have

$$\begin{aligned} &m_1 \approx 942\,\rho && (\bar{x}_1, \bar{y}_1) \approx (0, 19.3) \\ &m_2 = 700\rho && (\bar{x}_2, \bar{y}_2) = (0, -17.5) \\ &m_3 = 960\rho && (\bar{x}_3, \bar{y}_3) = (0, -59) \end{aligned}$$

4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.

Answer: Let $\left(\bar{x}_p, \bar{y}_p\right)$ denote the center of mass of the entire platform. By symmetry, $\bar{x}_p = 0$. To find $\bar{y}_p$, we need the total mass of the platform and the moment with respect to the *x*-axis.

We have

$$m_p = 2602\rho$$

and

$$\begin{aligned} M_{xp} &= \sum_{i=1}^{3} m_i \bar{y}_i = 942\rho(19.3) + 700\rho(-17.5) + 960\rho(-59) \\ &= (-50,709.4)\rho \end{aligned}$$

Then

$$\bar{y}_p = \frac{M_{xp}}{m_p} = \frac{(-50,709.4)\rho}{2602\rho} \approx -19.5$$

So

$$m_p \approx 2602\rho \qquad \left(\bar{x}_p, \bar{y}_p\right) \approx (0, -19.5)$$

Note that the center of gravity is over the canyon—about 15.5 feet from the edge!

5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3. Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?

Answer: The visitors' center weighs 2.2 million pounds, so its mass, m_v, is given by

$$m_v = \frac{2,200,000}{32} = 68,750 \text{ slugs}$$

We know $\left(\bar{x}_v, \bar{y}_v\right) = \left(\bar{x}_3, \bar{y}_3\right)$, so we have $\left(\bar{x}_v, \bar{y}_v\right) = \left(\bar{x}_3, \bar{y}_3\right) = (0, -59)$. Let $\left(\bar{x}_s, \bar{y}_s\right)$ denote the center of mass of the system. Then, as before, by symmetry, $\bar{x}_s = 0$, and we need to find $\bar{y}_s$. Then the mass of the system, m_s, is given by

$$m_s = m_p + m_v = 37,500 + 68,750 = 106,250 \text{ slugs}$$

Thus,

$$\begin{aligned} M_{xs} &= m_p \bar{y}_p + m_v \bar{y}_v \\ &= 37,500(-19.5) + 68,750(-59) \\ &= -731,250 - 4,056,250 \\ &= -4,787,500 \end{aligned}$$

Then $\bar{y}_s = \dfrac{M_{xs}}{m_s} = \dfrac{-4,787,500}{106,250} \approx -45.1$

In summary,

$$m_s = 106,250 \qquad \left(\bar{x}_s, \bar{y}_s\right) \approx (0, -45.1)$$

Notice that the center of mass is now on land, about 10 feet from the edge of the canyon.

6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Answer: Let m_t denote the mass of the tourists, and $(\bar{x}_t, \bar{y}_t)$ denote the tourists' center of mass. Then

$$m_t = \frac{800 \cdot 200}{32} = 5000 \text{ and } (\bar{x}_t, \bar{y}_t) = (0, 35)$$

Finally, let m_f denote the final mass of the entire system (skywalk, visitors' center plus tourists) and $(\bar{x}_f, \bar{y}_f)$ denote the center of mass of the entire system. As before, by symmetry, $\bar{x}_f = 0$.

Then

$$\begin{aligned} m_f &= m_s + m_t \\ &= 106,250 + 5,000 \\ &= 111,250 \end{aligned}$$

and

$$\begin{aligned} M_{xf} &= m_s \bar{y}_s + m_t \bar{y}_t \\ &= 106,250(-45.1) + 5,000(35) \\ &= -4,787,500 + 175,000 \\ &= -4,612,500 \end{aligned}$$

So we see that

$$\bar{y}_f = \frac{M_{xf}}{m_f} = \frac{-4,612,500}{111,250} \approx -41.5$$

In summary,

$$m_f = 111,250 \qquad (\bar{x}_f, \bar{y}_f) \approx (0, -41.5)$$

The center of mass is still on land, but it is closer to the edge.