Chapter 5
Integration
5.7. Integrals Resulting in Inverse Trigonometric Functions

Section Exercises

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

391. $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Answer: $\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$

392. $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

Answer: $\sin^{-1} x \Big|_{-1/2}^{1/2} = \frac{\pi}{3}$

393. $\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}}$

Answer: $\tan^{-1} x \Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$

394. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

Answer: $\tan^{-1} x \Big|_{-1/\sqrt{3}}^{\sqrt{3}} = \frac{\pi}{6}$

395. $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Answer: $\sec^{-1} x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$

396. $\int_1^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

Answer: $\sec^{-1} x \Big|_1^{2/\sqrt{3}} = \frac{\pi}{6}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

397. $\int \frac{dx}{\sqrt{9-x^2}}$

Answer: $\sin^{-1}\left(\frac{x}{3}\right)+C$

398. $\int \frac{dx}{\sqrt{1-16x^2}}$

Answer: $\frac{1}{4}\sin^{-1}(4x)+C$

399. $\int \frac{dx}{9+x^2}$

Answer: $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

400. $\int \frac{dx}{25+16x^2}$

Answer: $\frac{1}{20}\tan^{-1}\left(\frac{4x}{5}\right)+C$

401. $\int \frac{dx}{|x|\sqrt{x^2-9}}$

Answer: $\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)+C$

402. $\int \frac{dx}{|x|\sqrt{4x^2-16}}$

Answer: $\frac{1}{4}\sec^{-1}\left(\frac{x}{2}\right)+C$

403. Explain the relationship $-\cos^{-1}t+C=\int \frac{dt}{\sqrt{1-t^2}}=\sin^{-1}t+C.$ Is it true, in general, that $\cos^{-1}t=-\sin^{-1}t$?

Answer: $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$ So, $\sin^{-1}t=\frac{\pi}{2}-\cos^{-1}t.$ They differ by a constant.

404. Explain the relationship $\sec^{-1} t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1} t + C$. Is it true, in general, that $\sec^{-1} t = -\csc^{-1} t$?

Answer: $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$. They differ by a constant $\frac{\pi}{2}$

405. Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Answer: $\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

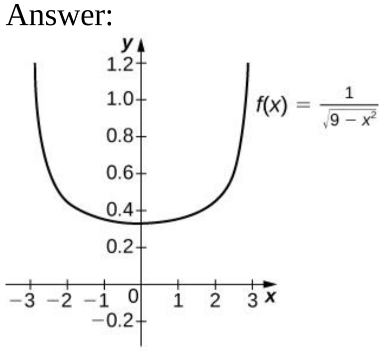
406. Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

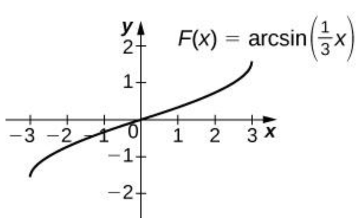
Answer: $\sqrt{t^2-1}$ is not defined as a real number when $|t| < 1$.

In the following exercises, solve for the antiderivative $\int f$ of *f* with $C = 0$, then use a calculator to graph *f* and the antiderivative over the given interval $[a, b]$. Identify a value of *C* such that adding *C* to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)\,dt$.

407. **[T]** $\int \frac{1}{\sqrt{9-x^2}}\,dx$ over $[-3, 3]$

Answer:

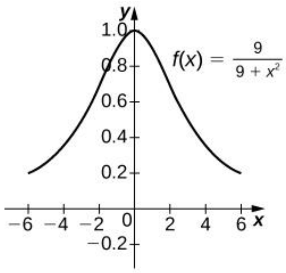


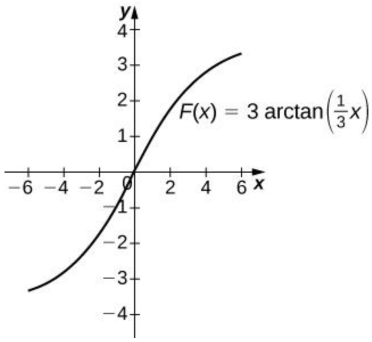


The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

408. **[T]** $\int \frac{9}{9+x^2}\,dx$ over $[-6,\ 6]$

Answer:

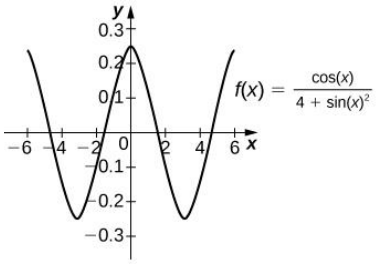


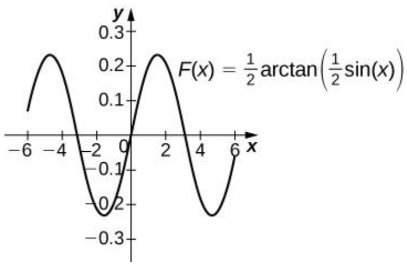


The antiderivative is $3\tan^{-1}\left(\frac{x}{3}\right)+C$. Taking $C=3\text{arctan}(2)$ recovers the definite integral.

409. **[T]** $\int \frac{\cos x}{4+\sin^2 x}\,dx$ over $[-6,\ 6]$

Answer:

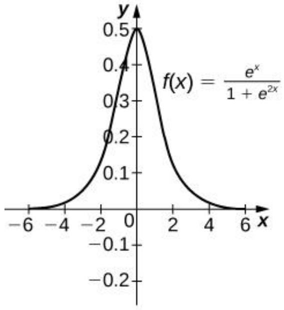


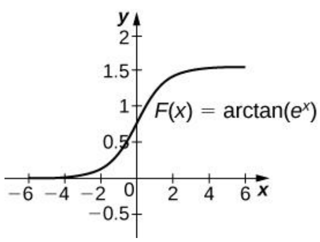


The antiderivative is $\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right)+C$. Taking $C=\frac{1}{2}\tan^{-1}\left(\frac{\sin(6)}{2}\right)$ recovers the definite integral.

410. **[T]** $\int \frac{e^x}{1+e^{2x}}dx$ over $[-6, 6]$

Answer:





The antiderivative is $\tan^{-1}(e^x)+C$. Taking $C = -\tan^{-1}(e^{-6})$ recovers the definite integral.

In the following exercises, compute the antiderivative using appropriate substitutions.

411. $\int \frac{\sin^{-1} t dt}{\sqrt{1-t^2}}$

Answer: $\frac{1}{2}(\sin^{-1} t)^2 + C$

412. $\int \frac{dt}{\sin^{-1} t\sqrt{1-t^2}}$

Answer: $\ln(\sin^{-1} t)+C$

413. $\int \frac{\tan^{-1}(2t)}{1+4t^2}dt$

Answer: $\frac{1}{4}(\tan^{-1}(2t))^2$

414. $\int \frac{t\tan^{-1}(t^2)}{1+t^4}dt$

Answer: $\frac{1}{4}(\tan^{-1}(t^2))^2$

415. $\displaystyle\int \frac{\sec^{-1}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}\,dt$

Answer: $\frac{1}{4}\left(\sec^{-1}\left(\frac{t}{2}\right)^2\right)+C$

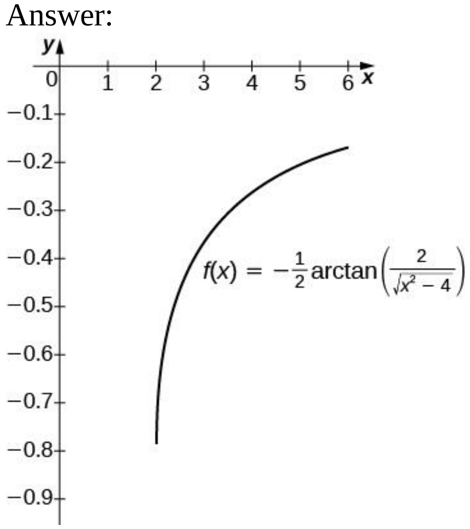
416. $\displaystyle\int \frac{t\sec^{-1}\left(t^2\right)}{t^2\sqrt{t^4-1}}\,dt$

Answer: $\frac{1}{4}\left(\sec^{-1}\left(t^2\right)\right)^2+C$

In the following exercises, use a calculator to graph the antiderivative $\int f$ with $C=0$ over the given interval $[a,\ b]$. Approximate a value of *C*, if possible, such that adding *C* to the antiderivative gives the same value as the definite integral $F(x)=\int_a^x f(t)\,dt$.

417. **[T]** $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$ over $[2,\ 6]$

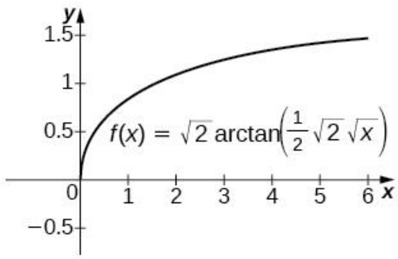
Answer:



The antiderivative is $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+C$. Taking $C=0$ recovers the definite integral over $[2,\ 6]$.

418. **[T]** $\int \frac{1}{(2x+2)\sqrt{x}}\,dx$ over $[0,\ 6]$

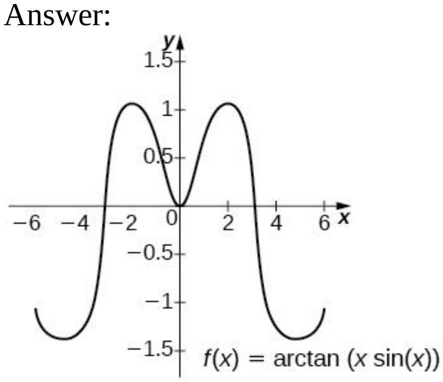
Answer:



The antiderivative is $\tan^{-1}\left(\sqrt{x}\right)+C$. Taking $C=0$ recovers the definite integral.

419. **[T]** $\int \frac{(\sin x + x\cos x)}{1+x^2\sin^2 x}\,dx$ over $[-6,\ 6]$

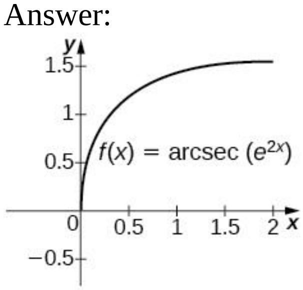
Answer:



The general antiderivative is $\tan^{-1}(x\sin x)+C$. Taking $C=-\tan^{-1}(6\sin(6))$ recovers the definite integral.

420. **[T]** $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}\,dx$ over $[0,\ 2]$

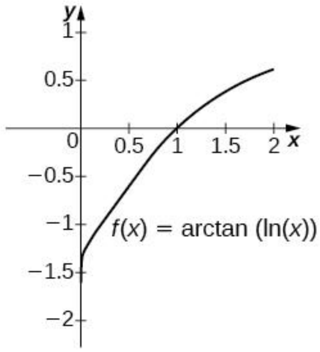
Answer:



The general antiderivative is $\sec^{-1}\left(e^{2x}\right)+C$. Taking $C=0$ appears to recover the definite integral.

421. **[T]** $\int \frac{1}{x + x\ln^2 x}$ over $[0, 2]$

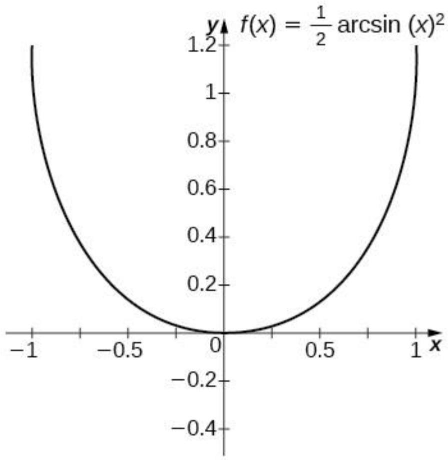
Answer:



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

422. **[T]** $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$ over $[-1, 1]$

Answer:



The general antiderivative is $\frac{1}{2}\sin^{-1}(x)^2 + C$. Taking $C = \frac{\pi^2}{8}$ recovers the definite integral.

In the following exercises, compute each integral using appropriate substitutions.

423. $\int \frac{e^t}{\sqrt{1 - e^{2t}}}\,dt$

Answer: $\sin^{-1}(e^t) + C$

424. $\displaystyle\int \frac{e^t}{1+e^{2t}}\,dt$

Answer: $\tan^{-1}\left(e^t\right)+C$

425. $\displaystyle\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

Answer: $\sin^{-1}\left(\ln t\right)+C$

426. $\displaystyle\int \frac{dt}{t\left(1+\ln^2 t\right)}$

Answer: $\tan^{-1}\left(\ln t\right)+C$

427. $\displaystyle\int \frac{\cos^{-1}\left(2t\right)}{\sqrt{1-4t^2}}\,dt$

Answer: $-\dfrac{1}{2}\left(\cos^{-1}\left(2t\right)\right)^2+C$

428. $\displaystyle\int \frac{e^t\cos^{-1}\left(e^t\right)}{\sqrt{1-e^{2t}}}\,dt$

Answer: $-\dfrac{1}{2}\left(\cos^{-1}\left(e^t\right)\right)^2+C$

In the following exercises, compute each definite integral.

429. $\displaystyle\int_0^{1/2} \frac{\tan\left(\sin^{-1} t\right)}{\sqrt{1-t^2}}\,dt$

Answer: $\dfrac{1}{2}\ln\left(\dfrac{4}{3}\right)$

430. $\displaystyle\int_{1/4}^{1/2} \frac{\tan\left(\cos^{-1} t\right)}{\sqrt{1-t^2}}\,dt$

Answer: $\ln 2$

431. $\displaystyle\int_0^{1/2} \frac{\sin\left(\tan^{-1} t\right)}{1+t^2}\,dt$

Answer: $1-\dfrac{2}{\sqrt{5}}$

432. $\displaystyle\int_0^{1/2} \frac{\cos\left(\tan^{-1} t\right)}{1+t^2}\,dt$

Answer: $\dfrac{1}{\sqrt{5}}$

433. For $A > 0,$ compute $I\left(A\right) = \displaystyle\int_{-A}^{A} \frac{dt}{1+t^2}$ and evaluate $\lim\limits_{a\to\infty} I\left(A\right),$ the area under the graph of $\dfrac{1}{1+t^2}$ on $\left[-\infty,\, \infty\right].$

Answer: $2\tan^{-1}\left(A\right) \to \pi$ as $A \to \infty$

434. For $1 < B < \infty,$ compute $I(B) = \displaystyle\int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim\limits_{B\to\infty} I(B),$ the area under the graph of $\dfrac{1}{t\sqrt{t^2-1}}$ over $[1,\infty).$

Answer: $\sec^{-1}(B) - \sec^{-1}(1) \to \dfrac{\pi}{2}$ as $B \to \infty$

435. Use the substitution $u = \sqrt{2}\cot x$ and the identity $1+\cot^2 x = \csc^2 x$ to evaluate $\displaystyle\int \frac{dx}{1+\cos^2 x}.$ (*Hint:* Multiply the top and bottom of the integrand by $\csc^2 x.$))

Answer: Using the hint, one has $\displaystyle\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}\,dx = \int \frac{\csc^2 x}{1+2\cot^2 x}\,dx.$ Set $u = \sqrt{2}\cot x.$ Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\dfrac{1}{\sqrt{2}}\displaystyle\int \frac{du}{1+u^2} = -\frac{1}{\sqrt{2}}\tan^{-1} u + C$ $= \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cot x\right) + C.$ If one uses the identity $\tan^{-1} s + \tan^{-1}\left(\dfrac{1}{s}\right) = \dfrac{\pi}{2},$ then this can also be written $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x}{\sqrt{2}}\right) + C.$

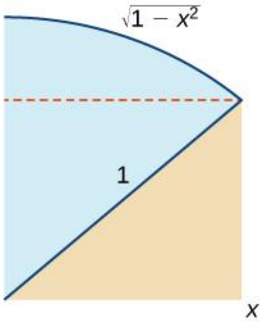
436. **[T]** Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g\left(x\right) = \left(1+4x^2\right)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

Answer: $x = \pm 0.763.$ The left endpoint estimate with $N = 100$ gives the area 1.770 and these decimals stay the same for $N = 500$

437. **[T]** Approximate the points at which the graphs of $f(x)=x^2-1$ and $f(x)=x^2-1$ intersect, and approximate the area between their graphs accurate to three decimal places.

Answer: $x\approx\pm1.13525.$ The left endpoint estimate with $N=100$ is 2.796 and these decimals persist for $N=500$.

438. Use the following graph to prove that $\int_0^x \sqrt{1-t^2}\,dt=\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x.$



Answer: The integral represents the area below the graph, which is the sum of the shaded areas. The triangle has area $1/2\,x\sqrt{1-x^2}$. The area above is the area of a wedge subtended by the angle with sine equal to *x*.

Chapter Review Exercises

True or False. **Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.**

439. If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)$. Use a graph to justify your answer.

Answer: False

440. $\int_a^b f(x)^2 dx = \int_a^b f(x)dx \int_a^b f(x)dx$

Answer: False

441. If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \le \int_a^b g(x)$.

Answer: True

442. All continuous functions have an antiderivative.

Answer: True

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

443. $y = 3x^2 - 2x + 1$ over $[-1, 1]$

Answer: $L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

444. $y = \ln(x^2 + 1)$ over $[0, e]$

Answer: $L_4 = 2.084,\ R_4 = 3.529,$ exact answer: 2.782

445. $y = x^2 \sin x$ over $[0, \pi]$

Answer: $L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

446. $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Answer: $L_4 = 5.989,\ R_4 = 6.177,$ exact answer: $\frac{14}{3} + \ln(4))$

Evaluate the following integrals.

447. $\int_{-1}^{1}\left(x^3-2x^2+4x\right)dx$

Answer: $-\frac{4}{3}$

448. $\int_0^4 \frac{3t}{\sqrt{1+6t^2}}\,dt$

Answer: $\frac{1}{2}\left(\sqrt{97}-1\right)$

449. $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

Answer: 1

450. $\int_0^{\pi/4} e^{\cos^2 x}\sin x\cos dx$

Answer: $\frac{1}{2}\left(e-\sqrt{e}\right)$

Find the antiderivative.

451. $\int \frac{dx}{(x+4)^3}$

Answer: $-\frac{1}{2(x+4)^2}+C$

452. $\int x\ln\left(x^2\right)dx$

Answer: $\frac{x^2}{2}\left(\ln\left(x^2\right)-1\right)+C$

453. $\int \frac{4x^2}{\sqrt{1-x^6}}\,dx$

Answer: $\frac{4}{3}\sin^{-1}\left(x^3\right)+C$

454. $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

Answer: $\frac{1}{2}\tan^{-1}\left(e^{2x}\right)+C$

Find the derivative.

455. $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1+x^2}}\,dx$

Answer: $\frac{\sin t}{\sqrt{1+t^2}}$

456. $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

Answer: $3x^2\sqrt{4-x^6}$

457. $\frac{d}{dx}\int_1^{\ln(x)} \left(4t+e^t\right)dt$

Answer: $4\frac{\ln x}{x}+1$

458. $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

Answer: $-e^{\cos^2 x}\sin x$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	–5,468,750
1990	–755,495
1995	–73,005
2000	–29,768
2005	–918
2010	–177

459. If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.
Answer: $6,328,113

460. The average cost per gigabyte of RAM can be approximated by the function $C(t)=8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US$. Find the average cost per gigabyte of RAM for 1980 to 2010.
Answer: $657,716

461. Find the average cost of 1GB RAM for 2005 to 2010.
Answer: $73.36

462. The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686,$ where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \le t \le 0.5.$ What is the total distance the bullet travels in 0.5 sec?

Answer: $\frac{19117}{24}\text{ ft},$ or 796.5 ft

463. What is the average velocity of the bullet for the first half-second?

Answer: $\frac{19117}{12}\text{ ft/sec},$ or 1593 ft/sec