Chapter 5
Integration
5.3. The Fundamental Theorem of Calculus

Section Exercises

144. Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

Answer: The average value of $v_1(t)-v_2(t)$ is zero, so there must be some *t* during the race when $v_1(t)=v_2(t)$.

145. Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

Answer: Yes. It is implied by the Mean Value Theorem for Integrals.

146. To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

Answer: The average speed can be measured at the tollbooth. If the average speed exceeds the speed limit, then the driver must have been speeding at some point.

147. Set $F(x)=\int_1^x (1-t)\,dt$. Find $F'(2)$ and the average value of F' over $[1,\ 2]$.

Answer: $F'(2)=-1$; average value of F' over $[1,\ 2]$ is $-1/2$.

In the following exercises, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

148. $\dfrac{d}{dx}\int_1^x e^{-t^2}\,dt$

Answer: e^{-x^2}

149. $\dfrac{d}{dx}\int_1^x e^{\cos t}\,dt$

Answer: $e^{\cos x}$

150. $\dfrac{d}{dx}\int_3^x \sqrt{9-y^2}\,dy$

Answer: $\sqrt{9-x^2}$

151. $\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}}$

Answer: $\frac{1}{\sqrt{16-x^2}}$

152. $\frac{d}{dx}\int_x^{2x} t\,dt$

Answer: $3x$

153. $\frac{d}{dx}\int_0^{\sqrt{x}} t\,dt$

Answer: $\sqrt{x}\frac{d}{dx}\sqrt{x}=\frac{1}{2}$

154. $\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}\,dt$

Answer: $\sqrt{1-\sin^2 x}\,\frac{d}{dx}\sin x=|\cos x|\cos x$

155. $\frac{d}{dx}\int_{\cos x}^{1} \sqrt{1-t^2}\,dt$

Answer: $-\sqrt{1-\cos^2 x}\,\frac{d}{dx}\cos x=|\sin x|\sin x$

156. $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt$

Answer: $\frac{1}{2\sqrt{x}}\frac{|x|}{1+x^2}$

157. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt$

Answer: $2x\frac{|x|}{1+x^2}$

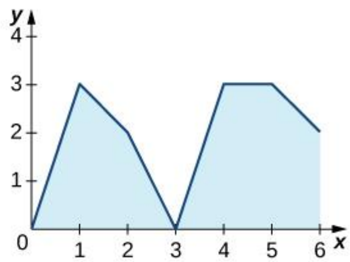
158. $\frac{d}{dx}\int_0^{\ln x} e^t\,dt$

Answer: $e^{\ln x}\frac{d}{dx}\ln x=1$

159. $\frac{d}{dx}\int_1^{e^x} \ln u^2\, du$

Answer: $\ln(e^{2x})\frac{d}{dx}e^x = 2xe^x$

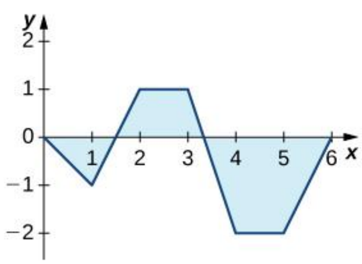
160. The graph of $y = \int_0^x f(t)\,dt,$ where *f* is a piecewise constant function, is shown here.



a. Over which intervals is *f* positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
b. What are the maximum and minimum values of *f*?
c. What is the average value of *f*?

Answer: a. *f* is positive over $[0,\ 1]$ and $[3,\ 4]$, negative over $[1,\ 3]$ and $[5,\ 6]$, and zero over $[4,\ 5]$. b. The maximum value is 3 and the minimum is –2. c. The average is $1/3$.

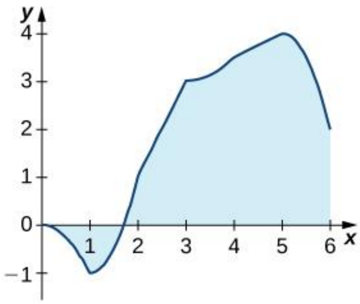
161. The graph of $y = \int_0^x f(t)\,dt,$ where *f* is a piecewise constant function, is shown here.



a. Over which intervals is *f* positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
b. What are the maximum and minimum values of *f*?
c. What is the average value of *f*?

Answer: a. *f* is positive over $[1,\ 2]$ and $[5,\ 6]$, negative over $[0,\ 1]$ and $[3,\ 4]$, and zero over $[2,\ 3]$ and $[4,\ 5]$. b. The maximum value is 2 and the minimum is –3. c. The average value is 0.

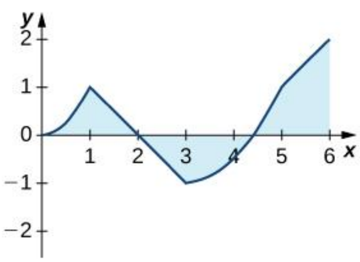
162. The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?
c. What is the average value of ℓ?

Answer: a. ℓ is positive over $[1,\ 5]$, negative over $[0,\ 1]$ and $[5,\ 6]$. b. It is increasing over $[1,\ 2]$ and $[3,\ 4]$, decreasing over $[0,\ 1]$ and $[5,\ 6]$, and constant over $[2,\ 3]$ and $[4,\ 5]$. c. The average value is $1/3$

163. The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?
c. What is the average value of ℓ?

Answer: a. ℓ is positive over $[0,\ 1]$ and $[3,\ 6]$, and negative over $[1,\ 3]$. b. It is increasing over $[0,\ 1]$ and $[3,\ 5]$, and it is constant over $[1,\ 3]$ and $[5,\ 6]$. c. Its average value is $\frac{1}{3}$.

In the following exercises, use a calculator to estimate the area under the curve by computing T_{10}, the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

164. **[T]** $y = x^2$ over $[0,\ 4]$

Answer: $T_{10} = 21.4400$, $\int_0^4 x^2 dx = \dfrac{64}{3}$

165. **[T]** $y = x^3 + 6x^2 + x - 5$ over $[-4,\ 2]$

Answer: $T_{10} = 49.08$, $\int_{-2}^{3} x^3 + 6x^2 + x - 5dx = 48$

166. **[T]** $y = \sqrt{x^3}$ over $[0,\ 6]$

Answer: $T_{10} = 35.376$ (rounded), $\int_0^6 x^{3/2} dx = \dfrac{72\sqrt{6}}{5} \approx 35.273$

167. **[T]** $y = \sqrt{x} + x^2$ over $[1,\ 9]$

Answer: $T_{10} = 260.836, \int_1^9 \left(\sqrt{x} + x^2\right) dx = 260$

168. **[T]** $\int (\cos x - \sin x)\, dx$ over $[0,\ \pi]$

Answer: $T_{10} = -1.984$ (rounded), $\int_0^{\pi} (\cos x - \sin x)\, dx = -2$

169. **[T]** $\int \dfrac{4}{x^2} dx$ over $[1,\ 4]$

Answer: $T_{10} = 3.058$, $\displaystyle\int_1^4 \frac{4}{x^2} dx = 3$

In the following exercises, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

170. $\int_{-1}^{2} \left(x^2 - 3x\right) dx$

Answer: $F(x) = \dfrac{x^3}{3} - \dfrac{3x^2}{2}$, $F(2) - F(1) = -\dfrac{3}{2}$

171. $\int_{-2}^{3} \left(x^2 + 3x - 5\right) dx$

Answer: $F(x) = \dfrac{x^3}{3} + \dfrac{3x^2}{2} - 5x$, $F(3) - F(-2) = -\dfrac{35}{6}$

172. $\int_{-2}^{3}(t+2)(t-3)\,dt$

Answer: $F(x)=\frac{t^3}{3}-\frac{t^2}{2}-6t$, $F(3)-F(-2)=-\frac{125}{6}$

173. $\int_{2}^{3}(t^2-9)(4-t^2)\,dt$

Answer: $F(x)=-\frac{t^5}{5}+\frac{13t^3}{3}-36t$, $F(3)-F(2)=\frac{62}{15}$

174. $\int_{1}^{2}x^9\,dx$

Answer: $F(t)=\frac{t^{10}}{10}$, $F(2)-F(1)=\frac{1023}{10}$

175. $\int_{0}^{1}x^{99}\,dx$

Answer: $F(x)=\frac{x^{100}}{100}$, $F(1)-F(0)=\frac{1}{100}$

176. $\int_{4}^{8}\left(4t^{5/2}-3t^{3/2}\right)dt$

Answer: $F(x)=\frac{8}{7}x^{7/2}-\frac{6}{5}x^{5/2}$, $F(8)-F(4)=\frac{64}{35}\left(556\sqrt{2}-59\right)$

177. $\int_{1/4}^{4}\left(x^2-\frac{1}{x^2}\right)dx$

Answer: $F(x)=\frac{x^3}{3}+\frac{1}{x}$, $F(4)-F\left(\frac{1}{4}\right)=\frac{1125}{64}$

178. $\int_{1}^{2}\frac{2}{x^3}\,dx$

Answer: $F(x)=-x^{-2}$, $F(2)-F(1)=\frac{3}{4}$

179. $\int_{1}^{4}\frac{1}{2\sqrt{x}}\,dx$

Answer: $F(x)=\sqrt{x}$, $F(4)-F(1)=1$

180. $\int_1^4 \frac{2-\sqrt{t}}{t^2}\,dt$

Answer: $F(x) = -\frac{2}{t} + 2t^{-1/2}$, $F(4) - F(1) = \frac{1}{2}$

181. $\int_1^{16} \frac{dt}{t^{1/4}}$

Answer: $F(x) = \frac{4}{3}t^{3/4}$, $F(16) - F(1) = \frac{28}{3}$

182. $\int_0^{2\pi} \cos\theta\, d\theta$

Answer: $F(x) = \sin x$, $F(2\pi) - F(0) = 0$

183. $\int_0^{\pi/2} \sin\theta\, d\theta$

Answer: $F(x) = -\cos x$, $F\left(\frac{\pi}{2}\right) - F(0) = 1$

184. $\int_0^{\pi/4} \sec^2\theta\, d\theta$

Answer: $F(x) = \tan\theta, F(\pi/4) - F(0) = 1$

185. $\int_0^{\pi/4} \sec\theta \tan\theta d\theta$

Answer: $F(\theta) = \sec\theta$, $F\left(\frac{\pi}{4}\right) - F(0) = \sqrt{2} - 1$

186. $\int_{\pi/3}^{\pi/4} \csc\theta \cot\theta\, d\theta$

Answer: $F(x) = -\csc x$, $F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{3}\right) = -\sqrt{2} + \frac{2\sqrt{3}}{3}$

187. $\int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta$

Answer: $F(x) = -\cot(x)$, $F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{4}\right) = 1$

188. $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

Answer: $F(x) = -\frac{1}{x} + \frac{1}{2x^2}$, $F(2) - F(1) = \frac{1}{8}$

189. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

Answer: $F(x) = -\frac{1}{x} + \frac{1}{2x^2}$, $F(-1) - F(-2) = \frac{7}{8}$

In the following exercises, use the evaluation theorem to express the integral as a function $F(x)$.

190. $\int_a^x t^2\, dt$

Answer: $F(x) = \frac{1}{3}\left(x^3 - a^3 \right)$

191. $\int_1^x e^t\, dt$

Answer: $F(x) = e^x - e$

192. $\int_0^x \cos t\, dt$

Answer: $F(x) = \sin x$

193. $\int_{-x}^x \sin t\, dt$

Answer: $F(x) = 0$

In the following exercises, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

194. $\int_{-2}^3 |x|\, dx$

Answer: $\int_{-2}^0 (-x)\, dx + \int_0^3 x\, dx = \frac{13}{2}$

195. $\int_{-2}^4 \left| t^2 - 2t - 3 \right| dt$

Answer: $\int_{-2}^{-1} \left(t^2 - 2t - 3 \right) dt - \int_{-1}^{3} \left(t^2 - 2t - 3 \right) dt + \int_3^4 \left(t^2 - 2t - 3 \right) dt = \frac{46}{3}$

196. $\int_0^{\pi} |\cos t|\, dt$

Answer: $\int_0^{\pi/2} \cos t dt - \int_{\pi/2}^{\pi} \cos t dt = 2$

197. $\int_{-\pi/2}^{\pi/2} |\sin t|\, dt$

Answer: $-\int_{-\pi/2}^{0} \sin t dt + \int_0^{\pi/2} \sin t dt = 2$

198. Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos\left(\frac{\pi t}{6}\right)+12.25$, with *t* given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t1 and t2, where $0 \le t_1 < t_2 < 12$, do the number of daylight hours equal the average number?

c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2.

d. Compute the mean hours of daylight in Seattle between t_1 and t_2, where $0 \le t_1 < t_2 < 12$, and then between t_2 and t_1, and show that the average of the two is equal to the average day length.

Answer: a. The average is 12.25 daylight hours. b. $t = 3$ and $t = 9$. c. $12.25 - \frac{3.75}{6}\int_3^9 \cos\left(\frac{\pi t}{6}\right) dt = 12.25 + 2.38$. d. $\int_9^3 \cdots = -\int_3^9 \cdots$, so the integral from 9 to 3 cancels the integral from 3 to 9 and we are left with the average value.

199. Suppose the rate of gasoline consumption over the course of a year in the United States can be modeled by a sinusoidal function of the form $\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\times 10^9$ gal/mo.

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?
b. What is the number of gallons of gasoline consumed in the United States in a year?
c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April ($t=3$) and the end of September ($t=9$).

Answer: a. The average is 11.21×10^9 since $\cos\left(\frac{\pi t}{6}\right)$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos\left(\frac{\pi t}{6}\right)=0$, when $t=3$, and when $t=9$. b. Total consumption is the average rate times duration: $11.21\times 12\times 10^9 = 1.35\times 10^{11}$ c. $10^9\left(11.21-\frac{1}{6}\int_3^9 \cos\left(\frac{\pi t}{6}\right)dt\right)=10^9\left(11.21+\frac{2}{\pi}\right)=11.84x10^9$

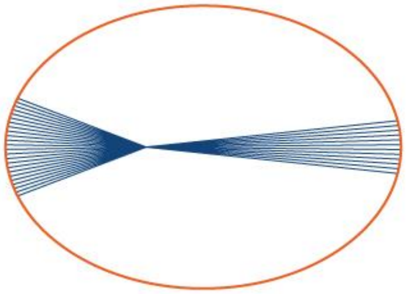
200. Explain why, if *f* is continuous over $[a,\ b]$, there is at least one point $c\in[a,\ b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$.

Answer: The average is between the maximum and the minimum, so the Intermediate Value Theorem applies.

201. Explain why, if *f* is continuous over $[a,\ b]$ and is not equal to a constant, there is at least one point $M\in[a,\ b]$ such that $f(M)=\frac{1}{b-a}\int_a^b f(t)\,dt$ and at least one point $m\in[a,\ b]$ such that $f(m)<\frac{1}{b-a}\int_a^b f(t)\,dt$.

Answer: If *f* is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a,b]$ by the extreme value theorem.

202. Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the *perihelion* (for Earth, it currently occurs around January 3) and the farthest point is called the *aphelion* (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



Answer: Earth is traveling fastest in its orbit when it is closest to the Sun because the area subtends a larger angle. Thus, it is moving fastest around January 3 and slowest around July 4.

203. A point on an ellipse with major axis length 2*a* and minor axis length 2*b* has the coordinates $(a\cos\theta, b\sin\theta)$, $0 \le \theta \le 2\pi$.

a. Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c\cos\theta$, where $c = \sqrt{a^2 - b^2}$.

b. Use these coordinates to show that the average distance $\overline{d}$ from a point on the ellipse to the focus at $(-c, 0)$, with respect to angle *θ*, is *a*.

Answer: a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2$; b.
$\overline{d} = \frac{1}{2\pi}\int_0^{2\pi}(a + 2c\cos\theta)\,d\theta = a$

204. As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

a. By placing the major axis along the *x*-axis, find the average distance from Earth to the Sun.

b. The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

Answer: a. Using a calculation similar to that in the previous exercise, with half the aphelion for *a* and half the perihelion for *b*, we can compute $a = 149,598,261 km$. b. Yes; it is equal to the average distance from the Earth to the Sun computed along the lines of the previous exercise.

205. The force of gravitational attraction between the Sun and a planet is $F(\theta)=\dfrac{GmM}{r^2(\theta)}$, where *m* is the mass of the planet, *M* is the mass of the Sun, *G* is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that *M*, *m*, and the ellipse parameters *a* and *b* (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G, m, M, a, b the average gravitational force between the Sun and the planet.

Answer: Mean gravitational force = $\dfrac{GmM}{2\pi}\displaystyle\int_0^{2\pi}\frac{1}{\left(a+\sqrt{a^2-b^2}\cos\theta\right)^2}\,d\theta$.

206. The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t)=A\cos(\omega t-\phi)$, where ϕ is a phase constant, ω is the angular frequency, and *A* is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

Answer: The average displacement is zero and the average velocity is zero. The average speed is $\dfrac{2A\omega}{\pi}$ and the average distance from rest is $\dfrac{2A}{\pi}$.

Student Project
A Parachutist in Free Fall

1. How long after she exits the aircraft does Julie reach terminal velocity?

Answer: We want to find the time when Julie's velocity reaches 176 ft/s. Solve

$$\begin{aligned} v(t) &= 176 \\ 32t &= 176 \\ t &= 5.5. \end{aligned}$$

Julie reaches terminal velocity after 5.5 seconds.

2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.

Answer: Based on our answer to question 1, Julia's velocity function is given by

$$v(t) = \begin{cases} 32t & 0 \le t \le 5.5 \\ 176 & 5.5 \le t \end{cases}$$

We can find the distance she has fallen by integrating this function.

$$\text{Distance fallen after 30 seconds} = \int_0^{5.5} 32t\,dt + \int_{5.5}^{30} 176\,dt$$

3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?

Answer: We want to find out how much time has elapsed when Julie pulls her ripcord. Let t_p denote this time. We know she pulls her ripcord after she has fallen 9,500 feet (from 12,500 feet to 3,000 feet), and we know we can figure out how far she has fallen by integrating her velocity function. So, we need to solve the following equation for t_p.

$$\int_0^{5.5} 32t\,dt + \int_{5.5}^{t_p} 176\,dt = 9500$$

Proceeding with the calculations, we get

$$\begin{aligned} \int_0^{5.5} 32t\,dt + \int_{5.5}^{t_p} 176\,dt &= 9500 \\ \left[16t^2\right]\Big|_0^{5.5} + \left[176t\right]\Big|_{5.5}^{t_p} &= 9500 \\ 16(30.25) - 0 + 176t_p - 176(5.5) &= 9500 \\ 484 + 176t_p - 968 &= 9500 \\ 176t_p &= 9984 \\ t_p &= 56.727. \end{aligned}$$

Julie is in free fall for 56.727 seconds—nearly a minute!

4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground.

Answer: Julie falls the last 2,600 feet at a speed of 16 ft/sec, so it takes her 162.5 seconds to reach the ground after she opens her canopy. Then Julie's total flight time is calculated by adding the time she spent in freefall (56.727 seconds, per question 2), the time it took her parachute to open (5 seconds), and the time it took her to reach the ground after her parachute opens (162.5 seconds). Total time of flight is 224.227 seconds, just short of 4 minutes.

5. How long does it take Julie to reach terminal velocity in this case?

Answer: Julie reaches terminal velocity after $\frac{220}{32} = 6.875$ seconds.

6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Answer: Let t_m denote the time Julie begins her reorientation maneuver. Then we want

$$\begin{aligned}
\int_0^{6.875} 32t\,dt + \int_{6.875}^{t_m} 220\,dt &= 8500 \\
\left[16t^2\right]\Big|_0^{6.875} + \left[220t\right]\Big|_{6.875}^{t_m} &= 8500 \\
16(47.266) - 0 + 220t_m - 220(6.875) &= 8500 \\
756.25 + 220t_m - 1512.5 &= 8500 \\
220t_m &= 9256.25 \\
t_m &= 42.074.
\end{aligned}$$

In this case, Julie is in free fall for 42.074 seconds prior to starting her reorientation maneuver.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air?

Answer: In this case, Julie reaches terminal velocity in just $\frac{44}{32} = 1.375$ seconds. If we again use t_p to denote the time when Julie pulls her ripcord, we want to solve

$$\begin{aligned}
\int_0^{1.375} 32t\,dt + \int_{1.375}^{t_p} 44\,dt &= 9500 \\
\left[16t^2\right]\Big|_0^{1.375} + \left[44t\right]\Big|_{1.375}^{t_p} &= 9500 \\
16(1.891) - 0 + 44t_p - 44(1.375) &= 9500 \\
30.25 + 44t_p - 60.5 &= 9500 \\
44t_p &= 9530.25 \\
t_p &= 216.597.
\end{aligned}$$

With a wingsuit on, Julie spends 216.597 seconds in freefall.