Chapter 5
Integration
5.6. Integrals Involving Exponential and Logarithmic Functions

Section Exercises
In the following exercises, compute each indefinite integral.

320. $\int e^{2x}\,dx$

Answer: $\frac{1}{2}e^{2x}+C$

321. $\int e^{-3x}\,dx$

Answer: $\frac{-1}{3}e^{-3x}+C$

322. $\int 2^{x}\,dx$

Answer: $\frac{2^x}{\ln 2}+C$

323. $\int 3^{-x}\,dx$

Answer: $-\frac{3^{-x}}{\ln 3}+C$

324. $\int \frac{1}{2x}\,dx$

Answer: $\frac{1}{2}\ln|x|+C$

325. $\int \frac{2}{x}\,dx$

Answer: $\ln\left(x^2\right)+C$

326. $\int \frac{1}{x^2}\,dx$

Answer: $-\frac{1}{x}+C$

327. $\int \frac{1}{\sqrt{x}}\,dx$

Answer: $2\sqrt{x}+C$

In the following exercises, find each indefinite integral by using appropriate substitutions.

328. $\int \frac{\ln x}{x}\,dx$

Answer: $\frac{1}{2}(\ln x)^2 + C$

329. $\int \frac{dx}{x(\ln x)^2}$

Answer: $-\frac{1}{\ln x} + C$

330. $\int \frac{dx}{x\ln x} \quad (x > 1)$

Answer: $\ln(\ln x) + C$

331. $\int \frac{dx}{x\ln x\ln(\ln x)}$

Answer: $\ln(\ln(\ln x)) + C$

332. $\int \tan\theta\,d\theta$

Answer: $\ln|\sec\theta| + C$

333. $\int \frac{\cos x - x\sin x}{x\cos x}\,dx$

Answer: $\ln(x\cos x) + C$

334. $\int \frac{\ln(\sin x)}{\tan x}\,dx$

Answer: $\frac{1}{2}(\ln(\sin(x)))^2 + C$

335. $\int \ln(\cos x)\tan x\,dx$

Answer: $-\frac{1}{2}(\ln(\cos(x)))^2 + C$

336. $\int xe^{-x^2}\,dx$

Answer: $\frac{-e^{-x^2}}{2} + C$

337. $\int x^2 e^{-x^3}\,dx$

Answer: $\frac{-e^{-x^3}}{3}+C$

338. $\int e^{\sin x}\cos x\,dx$

Answer: $e^{\sin x}+C$

339. $\int e^{\tan x}\sec^2 x\,dx$

Answer: $e^{\tan x}+C$

340. $\int e^{\ln x}\frac{dx}{x}$

Answer: $x+C$

341. $\int \frac{e^{\ln(1-t)}}{1-t}\,dt$

Answer: $t+C$

In the following exercises, verify by differentiation that $\int \ln x\,dx = x(\ln x - 1)+C$, then use appropriate changes of variables to compute the integral.

342. $\int x\ln x dx \left(\textit{Hint}\text{: } \int x\ln x dx = \frac{1}{2}\int x\ln\left(x^2\right)dx;\quad x>0 \right)$

Answer: $\frac{1}{4}x^2\left(\ln\left(x^2\right)-1\right)+C$

343. $\int x^2\ln^2 x dx$

Answer: $\frac{1}{9}x^3\left(\ln\left(x^3\right)-1\right)+C$

344. $\int \frac{\ln x}{x^2}dx$ (*Hint:* Set $u=\frac{1}{x}$.)

Answer: $\frac{1}{x}\left(\ln\left(\frac{1}{x}\right)-1\right)+C$

345. $\int \frac{\ln x}{\sqrt{x}}dx$ (*Hint:* Set $u=\sqrt{x}$.)

Answer: $2\sqrt{x}\left(\ln x-2\right)+C$

346. Write an integral to express the area under the graph of $y=\frac{1}{t}$ from $t=1$ to e^x and evaluate the integral.

Answer: $\int_1^{e^x}\frac{dt}{t}=\ln t\Big|_1^{e^x}=\ln\left(e^x\right)-\ln 1=x$

347. Write an integral to express the area under the graph of $y=e^t$ between $t=0$ and $t=\ln x$, and evaluate the integral.

Answer: $\int_0^{\ln x}e^t\,dt=e^t\Big|_0^{\ln x}=e^{\ln x}-e^0=x-1$

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

348. $\int\tan(2x)\,dx$

Answer: $-\frac{1}{2}\ln\cos(2x)+C$

349. $\int\frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}\,dx$

Answer: $-\frac{1}{3}\ln\left(\sin(3x)+\cos(3x)\right)$

350. $\int\frac{x\sin\left(x^2\right)}{\cos\left(x^2\right)}\,dx$

Answer: $-\frac{1}{2}\ln\left(\cos\left(x^2\right)\right)+C$

351. $\int x\csc\left(x^2\right)dx$

Answer: $-\frac{1}{2}\ln\left|\csc\left(x^2\right)+\cot\left(x^2\right)\right|+C$

352. $\int\ln(\cos x)\tan x\,dx$

Answer: $-\frac{1}{2}\left(\ln(\cos x)\right)^2+C$

353. $\int\ln(\csc x)\cot x\,dx$

Answer: $-\frac{1}{2}\left(\ln(\csc x)\right)^2+C$

354. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Answer: $\ln\left(e^x + e^{-x}\right) + C$

In the following exercises, evaluate the definite integral.

355. $\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3} dx$

Answer: $\frac{1}{3}\ln\left(\frac{26}{7}\right)$

356. $\int_0^{\pi/4} \tan x dx$

Answer: $\frac{\ln 2}{2}$

357. $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Answer: $\ln\left(\sqrt{3} - 1\right)$

358. $\int_{\pi/6}^{\pi/2} \csc x dx$

Answer: $-\ln\left(2 - \sqrt{3}\right)$

359. $\int_{\pi/4}^{\pi/3} \cot x dx$

Answer: $\frac{1}{2}\ln\frac{3}{2}$

In the following exercises, integrate using the indicated substitution.

360. $\int \frac{x}{x-100} dx$; $u = x - 100$

Answer: $x + 100\ln\left|x - 100\right| + C$

361. $\int \frac{y-1}{y+1} dy$; $u = y + 1$

Answer: $y - 2\ln\left|y + 1\right| + C$

362. $\displaystyle\int \frac{1-x^2}{3x-x^3}\,dx$; $u = 3x - x^3$

Answer: $\dfrac{1}{3}\ln\left(3x-x^3\right)+C$

363. $\displaystyle\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx$; $u=\sin x-\cos x$

Answer: $\ln\left|\sin x-\cos x\right|+C$

364. $\int e^{2x}\sqrt{1-e^{2x}}\,dx$; $u=e^{2x}$

Answer: $-\dfrac{1}{3}\left(1-e^{2x}\right)^{3/2}+C$

365. $\displaystyle\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx$; $u=\ln x$

Answer: $-\dfrac{1}{3}\left(1-\left(\ln x^2\right)\right)^{3/2}+C$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

366. **[T]** $y=e^x$ over $[0,\ 1]$

Answer: Exact solution: $e-1$, $R_{50}=1.736$. Since *f* is increasing, the right endpoint estimate is an overestimate.

367. **[T]** $y=e^{-x}$ over $[0,\ 1]$

Answer: Exact solution: $\dfrac{e-1}{e}$, $R_{50}=0.6258$. Since *f* is decreasing, the right endpoint estimate underestimates the area.

368. **[T]** $y=\ln(x)$ over $[1,\ 2]$

Answer: Exact solution: $\ln(4)-1$, $R_{50}=0.3932.$ Since *f* is increasing, the right endpoint estimate overestimates the area.

369. **[T]** $y=\dfrac{x+1}{x^2+2x+6}$ over $[0,\ 1]$

Answer: Exact solution: $\dfrac{2\ln(3)-\ln(6)}{2}$, $R_{50}=0.2033$. Since *f* is increasing, the right endpoint estimate overestimates the area.

370. **[T]** $y = 2^x$ over $[-1,\ 0]$

Answer: Exact solution: $\frac{1}{\ln(4)}$, $R_{50} = 0.7264$. Since *f* is increasing, the right endpoint estimate overestimates the area.

371. **[T]** $y = -2^{-x}$ over $[0,\ 1]$

Answer: Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since *f* is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values *a* and *b* by integrating.

372. $f(x) = \frac{\log_{10}(x)}{x}$; $a = 10$, $b = 100$

Answer: $\frac{3}{2}\ln(10)$

373. $f(x) = \frac{\log_2(x)}{x}$; $a = 32$, $b = 64$

Answer: $\frac{11}{2}\ln 2$

374. $f(x) = 2^{-x}$; $a = 1$, $b = 2$

Answer: $\frac{1}{\ln 16}$

375. $f(x) = 2^{-x}$; $a = 3$, $b = 4$

Answer: $\frac{1}{\ln(65{,}536)}$

376. Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Answer: $\frac{1}{2}\left(1 - e^{-25}\right)$

377. Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of *N* such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 1$ is, at most, 0.01.

Answer: $\int_N^{N+1} xe^{-x^2}\,dx = \frac{1}{2}\left(e^{-N^2} - e^{-(N+1)^2}\right)$. The quantity is less than 0.01 when $N = 2$.

378. Find the limit, as N tends to infinity, of the area under the graph of $f(x)=xe^{-x^2}$ between $x=0$ and $x=5$.

Answer: $\lim\limits_{N\to\infty}\frac{1}{2}\left(1-e^{-N^2}\right)=\frac{1}{2}$

379. Show that $\int_a^b \frac{dt}{t}=\int_{1/b}^{1/a}\frac{dt}{t}$ when $0<a\le b.$

Answer: $\int_a^b \frac{dx}{x}=\ln(b)-\ln(a)=\ln\left(\frac{1}{a}\right)-\ln\left(\frac{1}{b}\right)=\int_{1/b}^{1/a}\frac{dx}{x}$

380. Suppose that $f(x)>0$ for all x and that f and g are differentiable. Use the identity $f^g=e^{g\ln f}$ and the chain rule to find the derivative of f^g.

Answer: $\frac{d}{dt}e^{g\ln f}=e^{g\ln f}\left(\frac{dg}{dt}\ln f+\frac{g}{f}\frac{df}{dt}\right)$

381. Use the previous exercise to find the antiderivative of $h(x)=x^x(1+\ln x)$ and evaluate $\int_2^3 x^x(1+\ln x)\,dx.$

Answer: 23

382. Show that if $c>0,$ then the integral of $1/x$ from ac to bc $(0<a<b)$ is the same as the integral of $1/x$ from a to b.

Answer: $\int_{ac}^{ab}\frac{dx}{x}=\ln cb-\ln ac=\ln b-\ln a$

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x)=\int_1^x\frac{dt}{t},$ using properties of the definite integral and making no further assumptions.

383. Use the identity $\ln(x)=\int_1^x\frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right)=-\ln x.$

Answer: We may assume that $x>1,$ so $\frac{1}{x}<1.$ Then, $\int_1^{1/x}\frac{dt}{t}.$ Now make the substitution $u=\frac{1}{t},$ so $du=-\frac{dt}{t^2}$ and $\frac{du}{u}=-\frac{dt}{t},$ and change endpoints: $\int_1^{1/x}\frac{dt}{t}=-\int_1^x\frac{du}{u}=-\ln x.$

384. Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}\,dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

Answer: Set $u = \frac{t}{y}$ so $du = \frac{dt}{y}$ and $\int_1^{xy} \frac{1}{t} = \int_{u=1/y}^{x} \frac{ydu}{yu} = \int_{u=1/y}^{x} \frac{du}{u} = \ln x - \ln \frac{1}{y} = \ln x + \ln y.$

385. Use the identity $\ln x = \int_1^x \frac{dt}{x}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

Answer: This is a proof; therefore, no answer is provided.

386. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a+b) = E(a)E(b)$ for any real numbers a, b.

Answer: If $x = E(a)$ and $y = E(b)$, then $\ln xy = \ln\big(E(a)E(b)\big) = \ln E(a) + \ln E(b) = a + b$. Taking E of both sides and using the inverse relation gives $E(\ln xy) = E(a+b)$, but $E(\ln xy) = xy = E(a)E(b)$, so $E(a)E(b) = E(a+b)$ as claimed.

387. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Show that $E'(t) = E(t)$.

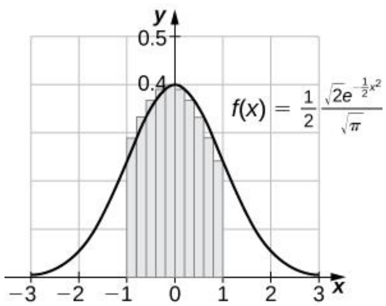
Answer: $x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x, and for such t we have $x = E(t)$, it follows that for any t, $E'(t) = E(t)$.

388. The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t}\,dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x. Show that for $k \ge 1$, $\left|S(2\pi k) - S\big(2\pi(k+1)\big)\right| \le \frac{1}{k(2k+1)\pi}$. (*Hint:* $\sin(t+\pi) = -\sin t$)

Answer: $\left|S\left(2\pi(k+1)\right)-S(2\pi k)\right| = \left|\int_{2\pi k}^{2\pi(k+1)} \frac{\sin t}{t}\,dt\right| = \left|\int_{2\pi k}^{2\pi(k+1)} \sin(t)\left(\frac{1}{t}-\frac{1}{t+\pi}\right)dt\right|$ using the hint. Since $\sin t \geq 0$ over $[0,\ \pi]$, and the denominator is increasing in *t*, the integral is bounded by $\frac{\pi}{(2k\pi)\left((2k+1)\pi\right)}\int_0^{\pi} \sin t\,dt = \frac{1}{k(2k+1)\pi}$, which was to be shown.

389. **[T]** The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The *standard normal distribution* in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the right endpoint estimates R_{10} and R_{100} of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^{2/2}}dx.$

Answer: $R_{10} = 0.6811,\ R_{100} = 0.6827$



390. **[T]** Compute the right endpoint estimates R_{50} and R_{100} of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}}e^{-(x-1)^2/8}.$

Answer: R_{50}=0.9544, $R_{100} = 0.9545$