Chapter 5
Integration
5.5. Substitution

Section Exercises

254. Why is u-substitution referred to as *change of variable*?
Answer: We are replacing the integration variable and differential—say, x and dx or t and dt with u and du.

255. If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'\left(h(x)\right)h'(x)$, should you take $u = g(x)$ or $u = h(x)$?
Answer: $u = h(x)$

In the following exercises, verify each identity using differentiation. Then, using the indicated *u*-substitution, identify *f* such that the integral takes the form $\int f(u)\,du$.

256. $\int x\sqrt{x+1}\,dx = \frac{2}{15}(x+1)^{3/2}(3x-2)+C$; $u = x+1$
Answer: $du = dx$; $f(u) = (u-1)\sqrt{u}$

257. For $x > 1$: $\int \frac{x^2}{\sqrt{x-1}}\,dx = \frac{2}{15}\sqrt{x-1}\left(3x^2+4x+8\right)+C$; $u = x-1$
Answer: $f(u) = \frac{(u+1)^2}{\sqrt{u}}$

258. $\int x\sqrt{4x^2+9}\,dx = \frac{1}{12}\left(4x^2+9\right)^{3/2}+C$; $u = 4x^2+9$
Answer: $du = 8x\,dx$; $f(u) = \frac{1}{8}\sqrt{u}$

259. $\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{4}\sqrt{4x^2+9}+C$; $u = 4x^2+9$
Answer: $du = 8x\,dx$; $f(u) = \frac{1}{8\sqrt{u}}$

260. $\int \frac{x}{(4x^2+9)^2}dx = -\frac{1}{8(4x^2+9)}$; $u = 4x^2+9$

Answer: $du = 8xdx$; $f(u) = \frac{1}{8u^2}$

In the following exercises, find the antiderivative using the indicated substitution.

261. $\int (x+1)^4 dx$; $u = x+1$

Answer: $\frac{1}{5}(x+1)^5 + C$

262. $\int (x-1)^5 dx$; $u = x-1$

Answer: $\frac{1}{6}(x-1)^6 + C$

263. $\int (2x-3)^{-7} dx$; $u = 2x-3$

Answer: $-\frac{1}{12(3-2x)^6} + C$

264. $\int (3x-2)^{-11} dx$; $u = 3x-2$

Answer: $-\frac{1}{30(2-3x)^{10}} + C$

265. $\int \frac{x}{\sqrt{x^2+1}}dx$; $u = x^2+1$

Answer: $\sqrt{x^2+1} + C$

266. $\int \frac{x}{\sqrt{1-x^2}}dx$; $u = 1-x^2$

Answer: $-\sqrt{1-x^2} + C$

267. $\int (x-1)(x^2-2x)^3 dx$; $u = x^2-2x$

Answer: $\frac{1}{8}(x^2-2x)^4 + C$

268. $\int (x^2-2x)(x^3-3x^2)^2\,dx$; $u=x^3-3x^2$

Answer: $\dfrac{(x^3-3x^2)^3}{9}+C$

269. $\int \cos^3\theta d\theta$; $u=\sin\theta$ (*Hint:* $\cos^2\theta = 1-\sin^2\theta$)

Answer: $\sin\theta-\dfrac{\sin^3\theta}{3}+C$

270. $\int \sin^3\theta d\theta$; $u=\cos\theta$ (*Hint:* $\sin^2\theta = 1-\cos^2\theta$)

Answer: $-\cos\theta+\dfrac{\cos^3\theta}{3}+C$

In the following exercises, use a suitable change of variables to determine the indefinite integral.

271. $\int x(1-x)^{99}\,dx$

Answer: $\dfrac{(1-x)^{101}}{101}-\dfrac{(1-x)^{100}}{100}+C$

272. $\int t(1-t^2)^{10}\,dt$

Answer: $\dfrac{1}{22}(t^2-1)^{11}+C$

273. $\int (11x-7)^{-3}\,dx$

Answer: $\int (11x-7)^{-3}dx=-\dfrac{1}{22(11x-7)^2}+C$

274. $\int (7x-11)^4\,dx$

Answer: $-\dfrac{1}{35}(11-7x)^5+C$

275. $\int \cos^3\theta\sin\theta d\theta$

Answer: $-\dfrac{\cos^4\theta}{4}+C$

276. $\int \sin^7 \theta \cos\theta d\theta$

Answer: $\frac{\sin^8 \theta}{8}+C$

277. $\int \cos^2(\pi t)\sin(\pi t)dt$

Answer: $-\frac{\cos^3(\pi t)}{3\pi}+C$

278. $\int \sin^2 x\cos^3 xdx$ (*Hint:* $\sin^2 x+\cos^2 x=1$)

Answer: $\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+C$

279. $\int t\sin\left(t^2\right)\cos\left(t^2\right)dt$

Answer: $-\frac{1}{4}\cos^2\left(t^2\right)+C$

280. $\int t^2\cos^2\left(t^3\right)\sin\left(t^3\right)dt$

Answer: $-\frac{1}{9}\cos^3\left(t^3\right)+C$

281. $\int \frac{x^2}{\left(x^3-3\right)^2}dx$

Answer: $-\frac{1}{3(x^3-3)}+C$

282. $\int \frac{x^3}{\sqrt{1-x^2}}dx$

Answer: $-\frac{1}{3}\sqrt{1-x^2}\left(x^2+2\right)+C$

283. $\int \frac{y^5}{\left(1-y^3\right)^{3/2}}dy$

Answer: $-\frac{2\left(y^3-2\right)}{3\sqrt{1-y^3}}$

284. $\int \cos\theta(1-\cos\theta)^{99}\sin\theta d\theta$

Answer: $\frac{\cos^{101}\theta}{101}-\frac{\cos^{100}\theta}{100}+C$

285. $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta d\theta$

Answer: $\frac{1}{33}(1-\cos^3\theta)^{11}+C$

286. $\int (\cos\theta-1)(\cos^2\theta-2\cos\theta)^3\sin\theta d\theta$

Answer: $-\frac{1}{8}(\cos^2\theta-2\cos\theta)^4+C$

287. $\int (\sin^2\theta-2\sin\theta)(\sin^3\theta-3\sin^2\theta)^3\cos\theta d\theta$

Answer: $\frac{1}{12}(\sin^3\theta-3\sin^2\theta)^4+C$

In the following exercises, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

288. **[T]** $y=3(1-x)^2$ over $[0,\ 2]$

Answer: $L_{50}=-2.0016$. The exact area is 2.

289. **[T]** $y=x(1-x^2)^3$ over $[-1,\ 2]$

Answer: $L_{50}=-8.5779.$ The exact area is $\frac{-81}{8}$

290. **[T]** $y=\sin x(1-\cos x)^2$ over $[0,\ \pi]$

Answer: $L_{50}=2.6654.$ The exact area is $\frac{8}{3}$

291. **[T]** $y=\frac{x}{(x^2+1)^2}$ over $[-1,\ 1]$

Answer: $L_{50}=-0.006399\ldots$ The exact area is 0.

In the following exercises, use a change of variables to evaluate the definite integral.

292. $\int_0^1 x\sqrt{1-x^2}dx$

Answer: $u=\left(1-x^2\right)$, $du=-2x\,dx$, $\int_0^1 \frac{1}{2}u^{1/2}du=\frac{1}{3}$

293. $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

Answer: $u=1+x^2$, $du=2x\,dx$, $\frac{1}{2}\int_1^2 u^{-1/2}du=\sqrt{2}-1$

294. $\int_0^2 \frac{t}{\sqrt{5+t^2}}dt$

Answer: $u=5+t^2$, $du=2t\,dt$, $\frac{1}{2}\int_5^9 u^{-1/2}du=3-\sqrt{5}$

295. $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}dt$

Answer: $u=1+t^3$, $du=3t^2$, $\frac{1}{3}\int_1^2 u^{-1/2}du=\frac{2}{3}\left(\sqrt{2}-1\right)$

296. $\int_0^{\pi/4}\sec^2\theta\tan\theta\,d\theta$

Answer: $u=\tan\theta$, $du=\sec^2\theta\,d\theta$, $\int_0^1 u\,du=\frac{1}{2}$

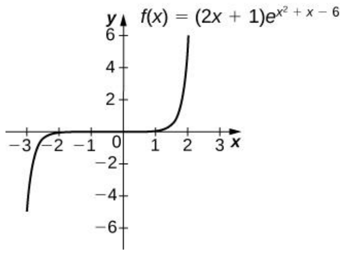
297. $\int_0^{\pi/4}\frac{\sin\theta}{\cos^4\theta}d\theta$

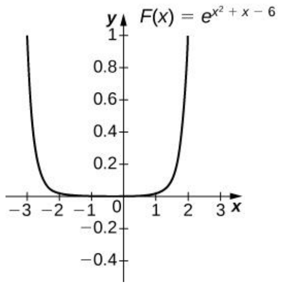
Answer: $u=\cos\theta$, $du=-\sin\theta\,d\theta$, $\int_{1/\sqrt{2}}^1 u^{-4}du=\frac{1}{3}\left(2\sqrt{2}-1\right)$

In the following exercises, evaluate the indefinite integral $\int f(x)\,dx$ with constant $C = 0$ using *u*-substitution . Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of *C* that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\,dt$, with *a* the left endpoint of the given interval.

298. **[T]** $\int (2x+1)e^{x^2+x-6}dx$ over $[-3,\ 2]$

Answer:

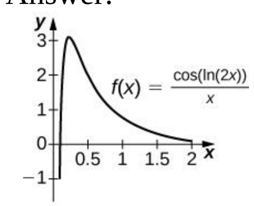


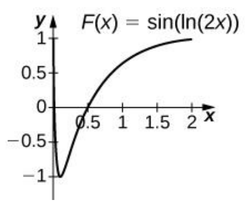


The antiderivative is e^{x^2+x-6} . $C = -1$ would give $e^{x^2+x-6} - 1 = 0$ at $x = -3$.

299. **[T]** $\int \frac{\cos(\ln(2x))}{x}dx$ on $[0,\ 2]$

Answer:

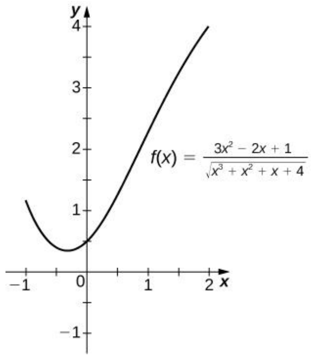


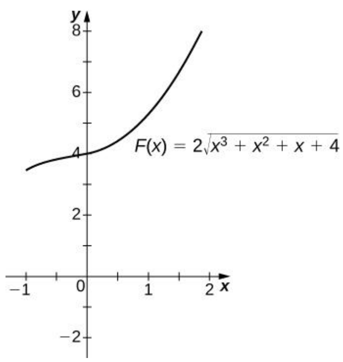


The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of *C* that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

300. **[T]** $\int \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx$ over $[-1,\ 2]$

Answer:

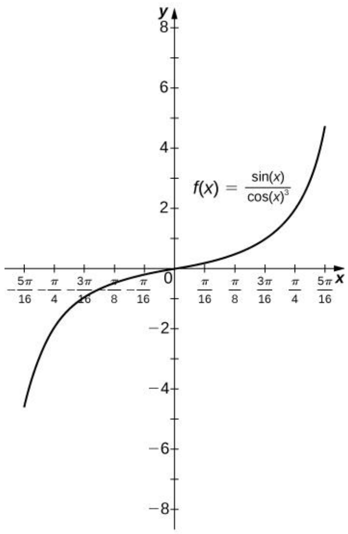


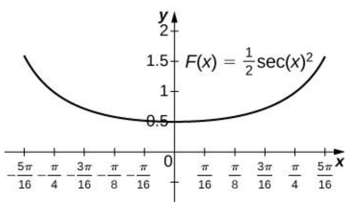


The antiderivative is $y = 2\sqrt{x^3+x^2+x+4}$. You should take $C = -2\sqrt{5}$ so that $F(-1) = 0$.

301. **[T]** $\int \frac{\sin x}{\cos^3 x}\,dx$ over $\left[-\frac{\pi}{3},\ \frac{\pi}{3}\right]$

Answer:

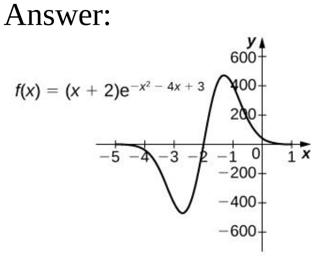


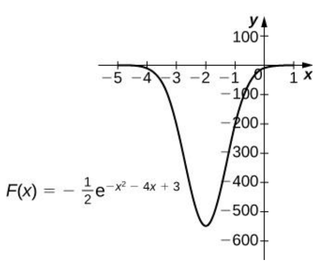


The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F\left(-\frac{\pi}{3}\right) = 0$.

302. **[T]** $\int (x+2)e^{-x^2-4x+3}dx$ over $[-5,1]$

Answer:

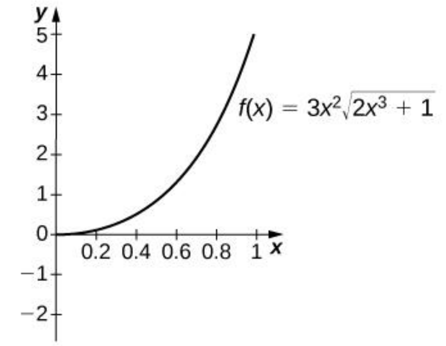


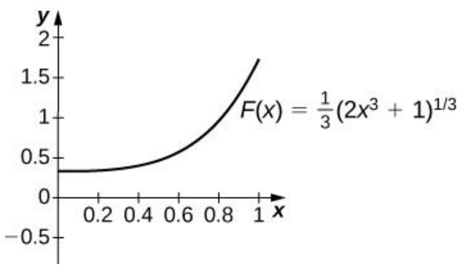


The antiderivative $y = -\frac{1}{2}e^{-x^2-4x+3}$. You should take $C = \frac{1}{2}e^{-2}$.

303. **[T]** $\int 3x^2\sqrt{2x^3+1}dx$ over $[0,1]$

Answer:





The antiderivative is $F(x)=\frac{1}{3}\left(2x^3+1\right)^{3/2}$. One should take $C=-\frac{1}{3}$.

304. If $h(a)=h(b)$ in $\int_a^b g'\left(h(x)\right)h(x)dx$, what can you say about the value of the integral?

Answer: The integral is zero.

305. Is the substitution $u=1-x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}dx$ okay? If not, why not?

Answer: No, because the integrand is discontinuous at $x=1$.

In the following exercises, use a change of variables to show that each definite integral is equal to zero.

306. $\int_0^\pi \cos^2(2\theta)\sin(2\theta)\,d\theta$

Answer: $u=\cos(2\theta)$; the integral becomes $-\frac{1}{2}\int_1^1 u^2\,du$.

307. $\int_0^{\sqrt{\pi}} t\cos\left(t^2\right)\sin\left(t^2\right)dt$

Answer: $u=\sin\left(t^2\right)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

308. $\int_0^1 (1-2t)\,dt$

Answer: $u = 1-2t$; the integral becomes $-\frac{1}{2}\int_1^{-1} u\,du = 0$ since the integrand is odd.

309. $\displaystyle\int_0^1 \frac{1-2t}{\left(1+\left(t-\frac{1}{2}\right)^2\right)}\,dt$

Answer: $u = \left(1+\left(t-\frac{1}{2}\right)^2\right)$; the integral becomes $-\displaystyle\int_{5/4}^{5/4} \frac{1}{u}\,du$.

310. $\displaystyle\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right)dt$

Answer: $u = t-\frac{\pi}{2}$; the integral becomes $\int_{-\pi/2}^{\pi/2} \sin\left(u^3\right)\cos u\,du$. The new integrand is odd with respect to *u*.

311. $\int_0^2 (1-t)\cos(\pi t)\,dt$

Answer: $u = 1-t$; the integral becomes

$\int_1^{-1} u\cos\left(\pi(1-u)\right)du$

$= \displaystyle\int_1^{-1} u\left[\cos\pi\cos\pi u - \sin\pi\sin\pi u\right]du$

$= -\displaystyle\int_1^{-1} u\cos\pi u\,du$

$= \displaystyle\int_{-1}^{1} u\cos\pi u\,du = 0$

since the integrand is odd.

312. $\int_{\pi/4}^{3\pi/4} \sin^2 t\cos t\,dt$

Answer: $u = \sin t$; the integral becomes $\int_{\sqrt{2}/2}^{\sqrt{2}/2} u^2\,du$.

313. Show that the average value of $f(x)$ over an interval $[a,\ b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c},\ \frac{b}{c}\right]$ for $c > 0$.

Answer: Setting $u = cx$ and $du = cdx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\,dx = \frac{c}{b-a}$ $\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)\,du$.

314. Find the area under the graph of $f(t) = \frac{t}{\left(1+t^2\right)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

Answer: $u = 1+t^2$, so $\int_0^x \frac{t}{\left(1+t^2\right)^a} = \frac{1}{2}\int_{u-1}^{1+x^2}\frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\Big|_1^{1+x^2} = \frac{1}{2(1-a)}\left(\left(1+x^2\right)^{1-a}-1\right)$. If $a > 1$, the limit is $\frac{1}{2(a-1)}$. If $a < 1$, the area tends to infinity as $x \to a$.

315. Find the area under the graph of $g(t) = \frac{t}{\left(1-t^2\right)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

Answer: $\int_0^x g(t)\,dt = \frac{1}{2}\int_{u=1-x^2}^{1}\frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\Big|_{u=1-x^2}^{1} = \frac{1}{2(1-a)}\left(1-\left(1-x^2\right)^{1-a}\right)$. As $x \to 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

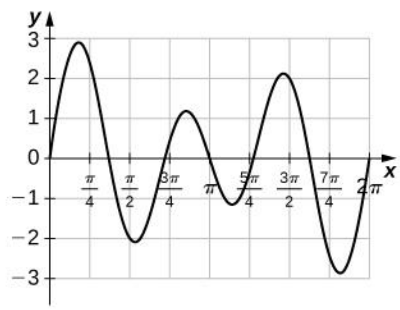
316. The area of a semicircle of radius 1 can be expressed as $\int_{-1}^{1}\sqrt{1-x^2}\,dx.$ Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

Answer: $x = -1$ corresponds to $t = \pi$; $x = 1$ corresponds to $t = 0$ and $dx = (-\sin t)\,dt$. Between π and zero, $\sin t \geq 0$, so $\sqrt{1-\cos^2 t} = \sin t$ and the integral becomes $\int_{t=-\pi}^{0}\sin t \times (-\sin t)\,dt = \int_0^{\pi}\sin^2 t\,dt$.

317. The area of the top half of an ellipse with a major axis that is the *x*-axis from $x=-a$ to a and with a minor axis that is the *y*-axis from $y=-b$ to b can be written as $\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}dx$. Use the substitution $x=a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

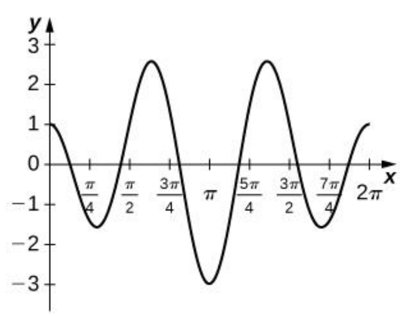
Answer: $\int_{t=\pi}^{0} b\sqrt{1-\cos^2 t}\times(-a\sin t)\,dt=\int_{t=0}^{\pi} ab\sin^2 t\,dt$

318. **[T]** The following graph is of a function of the form $f(t)=a\sin(nt)+b\sin(mt)$. Estimate the coefficients *a* and *b*, and the frequency parameters *n* and *m*. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.



Answer: $f(t)=\sin(2t)+2\sin(3t)$; $\int_0^{\pi}(\sin(2t)+2\sin(3t))\,dt=\frac{4}{3}$

319. **[T]** The following graph is of a function of the form $f(x)=a\cos(nt)+b\cos(mt)$. Estimate the coefficients *a* and *b* and the frequency parameters *n* and *m*. Use these estimates to approximate $\int_0^{\pi} f(t)\,dt$.



Answer: $f(t)=2\cos(3t)-\cos(2t)$; $\int_0^{\pi/2}(2\cos(3t)-\cos(2t))=-\frac{2}{3}$