Chapter 5
Integration
5.1. Approximating Areas

Section Exercises

1. State whether the given sums are equal or unequal.

   a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

   b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

   c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

   d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

Answer: a. They are equal; both represent the sum of the first 10 whole numbers. b. They are equal; both represent the sum of the first 10 whole numbers. c. They are equal by substituting $j = i-1$. d. They are equal; the first sum factors the terms of the second.

In the following exercises, use the rules for sums of powers of integers to compute the sums.

2. $\sum_{i=5}^{10} i$

Answer: $55-10=45$

3. $\sum_{i=5}^{10} i^2$

Answer: $385-30=355$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. In the following exercises, compute the sums.

4. $\sum_{i=1}^{100} (a_i + b_i)$

Answer: $15-12=3$

5. $\sum_{i=1}^{100} (a_i - b_i)$

Answer: $15-(-12)=27$

6. $\sum_{i=1}^{100} \left(3a_i - 4b_i\right)$

Answer: $3(15) - 4(-12) = 93$

7. $\sum_{i=1}^{100} \left(5a_i + 4b_i\right)$

Answer: $5(15) + 4(-12) = 27$

In the following exercises, use summation properties and formulas to rewrite and evaluate the sums.

8. $\sum_{k=1}^{20} 100\left(k^2 - 5k + 1\right)$

Answer: $100\left(\sum_{k=1}^{20} k^2 - 5\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1\right) = 184{,}000$

9. $\sum_{j=1}^{50} \left(j^2 - 2j\right)$

Answer: $\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40{,}375$

10. $\sum_{j=11}^{20} \left(j^2 - 10j\right)$

Answer: $\sum_{j=11}^{20} j^2 - 102\sum_{j=11}^{20} j = \left[\frac{(20)(21)(41)}{6} - 5(20)(21)\right] - \left[\frac{(10)(11)(21)}{9} - 5(10)(11)\right] = 935$

11. $\sum_{k=1}^{25} \left[(2k)^2 - 100k\right]$

Answer: $4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{6} - 50(25)(26) = -10{,}400$

Let L_n denote the left-endpoint sum using *n* subintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

12. L_4 for $f(x) = \frac{1}{x-1}$ on $[2,\ 3]$

Answer: $L_4 = \frac{319}{420}$

13. R_4 for $g(x) = \cos(\rho x)$ on $[0,\ 1]$

Answer: $R_4 = -0.25$

14. L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2,\ 5]$

Answer: $L_6 = 0.5972$

15. R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2,\ 5]$

Answer: $R_6 = 0.372$

16. R_4 for $\frac{1}{x^2+1}$ on $[-2,\ 2]$

Answer: $R_4 = 2.20$

17. L_4 for $\frac{1}{x^2+1}$ on $[-2,\ 2]$

Answer: $L_4 = 2.20$

18. R_4 for $x^2 - 2x + 1$ on $[0,\ 2]$

Answer: $R_4 = 0.6875$

19. L_8 for $x^2 - 2x + 1$ on $[0,\ 2]$

Answer: $L_8 = 0.6875$

20. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2,\ 2]$. Compute their average value and compare it with the area under the graph of *f*.

Answer: $L_4 = 4.0 = R_4$. The graph of *f* is a triangle of area 4.

21. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0,\ 6]$. Compute their average value and compare it with the area under the graph of *f*.

Answer: $L_6 = 9.000 = R_6$. The graph of *f* is a triangle with area 9.

22. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2,\ 2]$ and compare their values.

Answer: $L_4 = 5.4641 = R_4$. The graph of *f* is a semicircle of area $2\pi \approx 6.28$.

23. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0, 6]$ and compare their values.

Answer: $L_6 = 13.12899 = R_6$. They are equal.

Express the following endpoint sums in sigma notation but do not evaluate them.

24. L_{30} for $f(x) = x^2$ on $[1,\ 2]$

Answer: $L_{30} = \frac{1}{30} \sum_{i=1}^{30} \left(1 + \frac{i-1}{30}\right)^2$

25. L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2,\ 2]$

Answer: $L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - \left(-2 + 4\frac{(i-1)}{10}\right)}$

26. R_{20} for $f(x) = \sin x$ on $[0,\ \pi]$

Answer: $R_{20} = \frac{\pi}{20} \sum_{i=1}^{20} \sin\left(\pi \frac{i}{20}\right)$

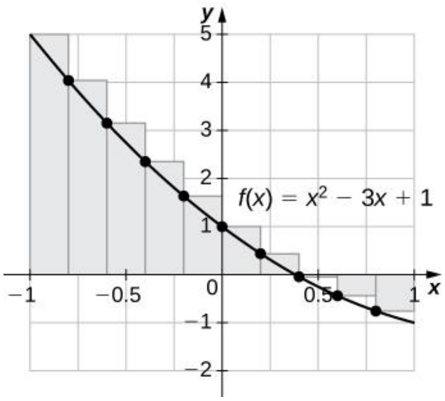
27. R_{100} for $\ln x$ on $[1,\ e]$

Answer: $R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} \ln\left(1 + (e-1)\frac{i}{100}\right)$

In the following exercises, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

28. **[T]** L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

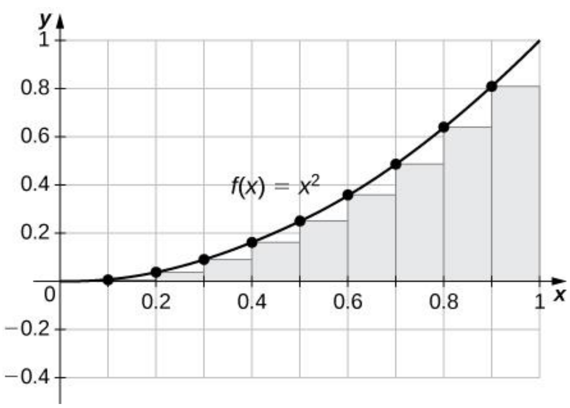
Answer:



$L_{100} = 2.7628$, $R_{100} = 2.6068$. The plot shows that the left endpoint sum is an overestimate because the function is decreasing. Similarly, the right Riemann sum is an underestimate. Thus, the area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

29. **[T]** L_{100} and R_{100} for $y = x^2$ on the interval $[0,1]$

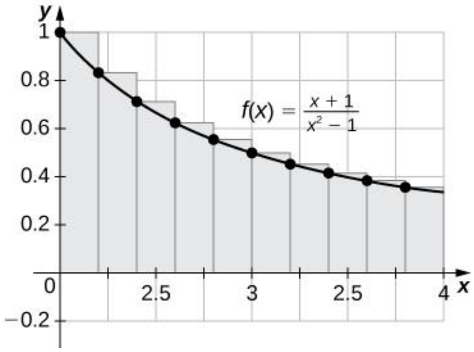
Answer:



$R_{100} = 0.33835$, $L_{100} = 0.32835$. The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

30. **[T]** L_{50} and R_{50} for $y=\dfrac{x+1}{x^2-1}$ on the interval $[2,\ 4]$

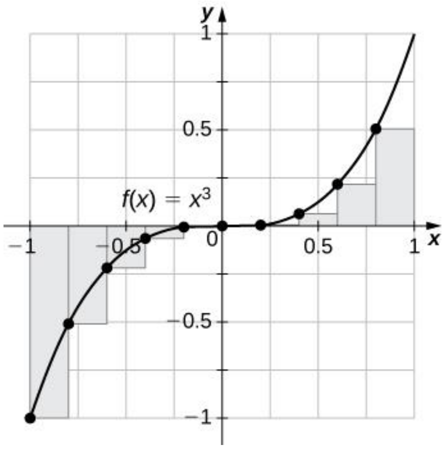
Answer:



$L_{50}=1.1121$, $R_{50}=1.0854$. The plot shows that the left Riemann sum overestimates the area because the function is decreasing. Similarly, the right endpoint sums are underestimates. The area lies between the left and right endpoint approximations. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

31. **[T]** L_{100} and R_{100} for $y=x^3$ on the interval $[-1,\ 1]$

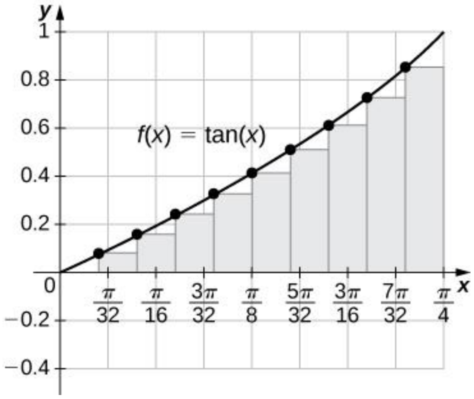
Answer:



$L_{100}=-0.02$, $R_{100}=0.02$. The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

32. **[T]** L_{50} and R_{50} for $y = \tan(x)$ on the interval $\left[0, \dfrac{\pi}{4}\right]$

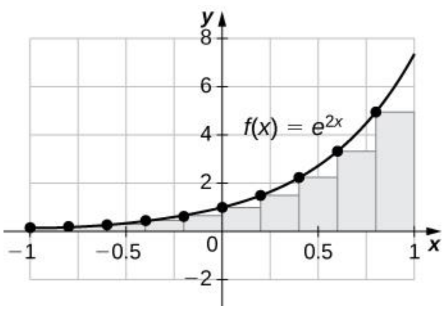
Answer:



$L_{50} = 0.3387$, $R_{50} = 0.3544$. The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

33. **[T]** L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Answer:



$L_{100} = 3.555$, $R_{100} = 3.670$. The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

34. Let t_j denote the time that it took Tejay van Garteren to ride the *j*th stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

Answer: The sum represents the total time it took van Garteren to complete the 2014 Tour de France.

35. Let r_j denote the total rainfall in Portland on the *j*th day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Answer: The sum represents the cumulative rainfall in January 2009.

36. Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j-1$ to day *j* in Fargo, North Dakota, on the *j*th day of the year. Interpret $d_1+\sum_{j=2}^{365}\delta_j$.

Answer: The sum represents the total number of hours of daylight on the 365th day of the year in Fargo.

37. To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Answer: The total mileage is $7\times\sum_{i=1}^{25}\left(1+\frac{(i-1)}{10}\right)=7\times 25+\frac{7}{10}\times 12\times 25=385\text{ mi}$.

38. The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Average Annual Atmospheric CO_2 Increase, 1964–2013

Decade	**Ppm/y**
1964–1973	1.07
1974–1983	1.34
1984–1993	1.40
1994–2003	1.87
2004–2013	2.07

Answer: Add the rates and multiply by 10 to get 77.5 ppm.

39. The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Approximate 20-Year Sea Level Increases, 1870–1990

Starting Year	**20-Year Change**
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Answer: Add the numbers to get 8.1-in. net increase.

40. The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was \$2.60, what was the average price of a gallon of gas in 1950?

Approximate 10-Year Gas Price Increases, 1950–2000

Starting Year	**10-Year Change**
1950	0.03
1960	0.05
1970	0.86
1980	–0.03
1990	0.29
2000	1.12

Answer: Subtract the increase of \$2.32 from \$2.60 to get 28¢ in 1950.

41. The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

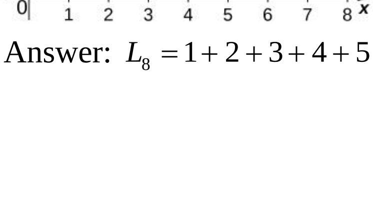
Annual Percentage Growth of U.S. Population, 2000–2009

Year	**% Change/Year**
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

(*Hint:* To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)
Answer: 309,389,957

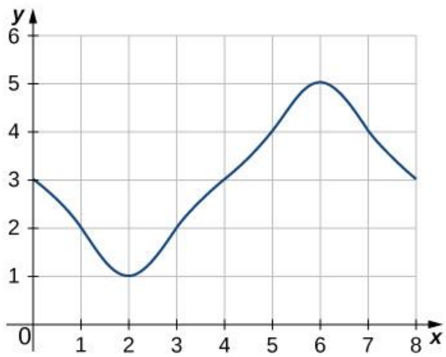
In the following exercises, estimate the areas under the curves by computing the left Riemann sums, L_8.

42.



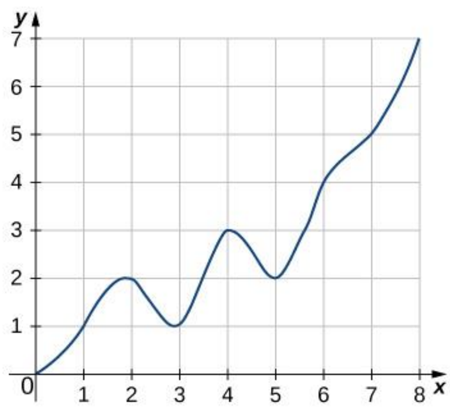
Answer: $L_8 = 1+2+3+4+5+4+3+2 = 24$

43.



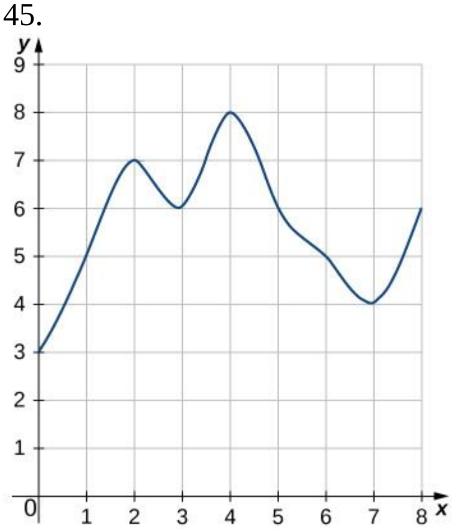
Answer: $L_8 = 3 + 2 + 1 + 2 + 3 + 4 + 5 + 4 = 24$

44.



Answer: $L_8 = 0 + 1 + 2 + 1 + 3 + 2 + 4 + 5 = 18$

45.



Answer: $L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$

46. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1-x^2}$ on $[-1,1]$.

Answer: $L_{10} \approx 1.5185$, $L_{30} \approx 1.5606$, $L_{50} \approx 1.5660$

47. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \frac{1}{\sqrt{1+x^2}}$ on $[-1,1]$.

Answer: $L_{10} \approx 1.7604$, $L_{30} \approx 1.7625$, $L_{50} \approx 1.76265$

48. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π.

Answer: $L_{10} \approx 3.141592654$, $L_{30} \approx 3.141592654$, $L_{50} \approx 3.141592654$. L_{10} is a lucky guess; L_{50} is an accurate estimate.

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?

49. **[T]** $y = \cos(\pi x)$ on the interval $[0, 1]$

Answer: $R_1 = -1$, $L_1 = 1$, $R_{10} = -0.1$, $L_{10} = 0.1$, $L_{100} = 0.01$, and $R_{100} = -0.1$. By symmetry of the graph, the exact area is zero.

50. **[T]** $y = 3x + 2$ on the interval $[3, 5]$

Answer: $R_1 = 34$, $L_1 = 22$, $R_{10} = 28.6$, $L_{10} = 27.4$, $R_{100} = 28.06$, and $L_{100} = 27.94$. The graph is a trapezoid with area 28.

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

51. **[T]** $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

Answer: $R_1 = 0$, $L_1 = 0$, $R_{10} = 2.4499$, $L_{10} = 2.4499$, $R_{100} = 2.1365$, $L_{100} = 2.1365$

52. **[T]** $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

Answer: $R_1 = 0.6931$, $L_1 = 0$, $R_{10} = 0.4205$, $L_{10} = 0.3512$, $R_{100} = 0.3898$, $L_{100} = 0.3828$

53. Explain why, if $f(a)\ge 0$ and *f* is increasing on $[a,\ b]$, that the left endpoint estimate is a lower bound for the area below the graph of *f* on $[a,\ b]$.

Answer: If $[c,\ d]$ is a subinterval of $[a,\ b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d-c)$. But, $f(c)\le f(x)$ for $c\le x\le d$, so the area under the graph of *f* between *c* and *d* is $f(c)(d-c)$ plus the area below the graph of *f* but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of *f* on $[a,\ b]$.

54. Explain why, if $f(b)\ge 0$ and *f* is decreasing on $[a,\ b]$, that the left endpoint estimate is an upper bound for the area below the graph of *f* on $[a,\ b]$.

Answer: If $[c,d]$ is a subinterval of $[a,\ b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d-c)$. But, $f(x)\ge f(x)$ for $c\le x\le d$, so the area under the graph of *f* between *c* and *d* is $f(c)(d-c)$ less the area above the graph of *f* but below the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left endpoint sum is greater than or equal to the area below the graph of *f* on $[a,\ b]$.

55. Show that, in general, $R_N-L_N=(b-a)\times\dfrac{f(b)-f(a)}{N}$.

Answer: $L_N=\dfrac{b-a}{N}\sum_{i=1}^{N}f\left(a+(b-a)\dfrac{i-1}{N}\right)=\dfrac{b-a}{N}\sum_{i=0}^{N-1}f\left(a+(b-a)\dfrac{i}{N}\right)$ and $R_N=\dfrac{b-a}{N}\sum_{i=1}^{N}f\left(a+(b-a)\dfrac{i}{N}\right)$. The left sum has a term corresponding to $i=0$ and the right sum has a term corresponding to $i=N$. In R_N-L_N, any term corresponding to $i=1,2,\ldots,N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N-L_N=\dfrac{b-a}{N}\left(f\left(a+(b-a)\right)\dfrac{N}{N}\right)-\left(f(a)+(b-a)\dfrac{0}{N}\right)=\dfrac{b-a}{N}\left(f(b)-f(a)\right)$.

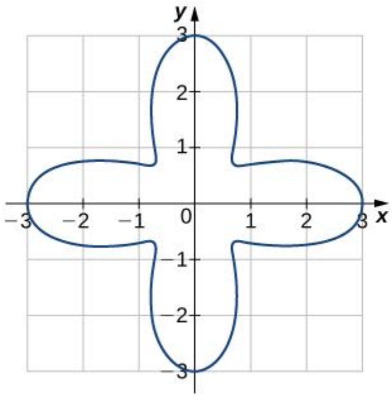
56. Explain why, if *f* is increasing on $[a,\ b]$, the error between either L_N or R_N and the area *A* below the graph of *f* is at most $(b-a)\dfrac{f(b)-f(a)}{N}$.

Answer: We can see that if *f* is increasing on $[a,\ b]$, then $L_N\le A\le R_N$.We know that $R_N-L_N=(b-a)\times\dfrac{f(b)-f(a)}{N}$. Since *A* is between the lower and upper sums, the difference between R_N and *A* is no bigger than that between L_N and *A*, and the desired inequality follows.

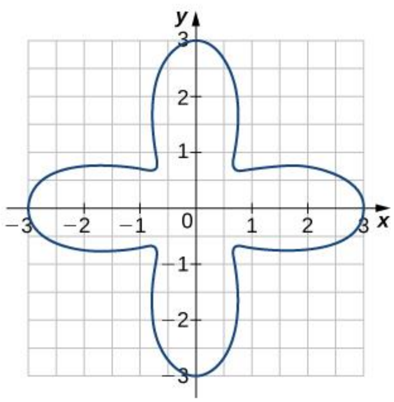
57. For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares *enclosed completely* by the curve.

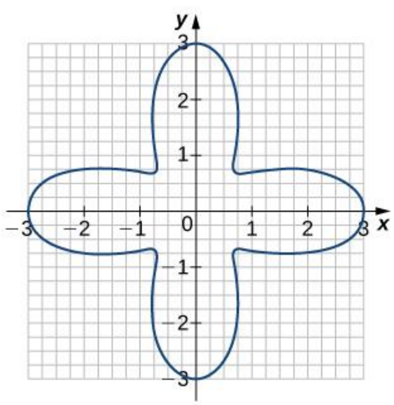
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares *enclosed partially* by the curve.



Graph 1



Graph 2



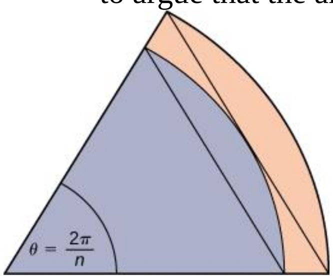
Graph 3

Answer: Graph 1: a. $L(A)=0$, $B(A)=20$; b. $U(A)=20$. Graph 2: a. $L(A)=9$; b. $B(A)=11$, $U(A)=20$. Graph 3: a. $L(A)=11.0$; b. $B(A)=4.5$, $U(A)=15.5$.

58. In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

Answer: When subdivided, each previous square is divided into four squares each of the same area. If the curve did not pass through one of the larger squares, then it will not pass through any of the subsquares. Thus, the enclosed area gets no smaller and the enclosing area gets no larger.

59. A unit circle is made up of *n* wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin\left(\frac{\pi}{n}\right)$. The base of the outer triangle is $B=\cos\left(\frac{\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$ and the height is $H=B\sin\left(\frac{2\pi}{n}\right)$. Use this information to argue that the area of a unit circle is equal to *π*.



Answer: Let *A* be the area of the unit circle. The circle encloses *n* congruent triangles each of area $\frac{\sin\left(\frac{2\pi}{n}\right)}{2}$, so $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\le A$. Similarly, the circle is contained inside *n* congruent triangles each of area $\frac{BH}{2}=\frac{1}{2}\left(\cos\left(\frac{\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)\right)\sin\left(\frac{2\pi}{n}\right)$, so

$A\le\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left(\cos\left(\frac{\pi}{n}\right)\right)+\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$. As $n\to\infty$, $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)=\frac{\pi\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)}\to\pi$, so we conclude $\pi\le A$. Also, as $n\to\infty$, $\cos\left(\frac{\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)\to 1$, so we also have $A\le\pi$. By the squeeze theorem for limits, we conclude that $A=\pi$.