Chapter 5
Integration
5.2. The Definite Integral

Section Exercises
In the following exercises, express the limits as integrals.

60. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^*\right)\Delta x$ over $[1,3]$

Answer: $\int_1^3 x dx$

61. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(5\left(x_i^*\right)^2 - 3\left(x_i^*\right)^3\right)\Delta x$ over $[0,2]$

Answer: $\int_0^2 \left(5x^2 - 3x^3\right) dx$

62. $\lim_{n\to\infty} \sum_{i=1}^{n} \sin^2\left(2\pi x_i^*\right)\Delta x$ over $[0,1]$

Answer: $\int_0^1 \sin^2\left(2\pi x\right) dx$

63. $\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2\left(2\pi x_i^*\right)\Delta x$ over $[0,1]$

Answer: $\int_0^1 \cos^2\left(2\pi x\right) dx$

In the following exercises, given L_n or R_n as indicated, express their limits as $n \to \infty$ as definite integrals, identifying the correct intervals.

64. $L_n = \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n}$

Answer: $\int_0^1 x\, dx$

65. $R_n = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}$

Answer: $\int_0^1 x\, dx$

66. $L_n = \frac{2}{n}\sum_{i=1}^{n}\left(1+2\frac{i-1}{n}\right)$

Answer: $\int_1^3 x\,dx$

67. $R_n = \frac{3}{n}\sum_{i=1}^{n}\left(3+3\frac{i}{n}\right)$

Answer: $\int_3^6 x\,dx$

68. $L_n = \frac{2\pi}{n}\sum_{i=1}^{n}2\pi\frac{i-1}{n}\cos\left(2\pi\frac{i-1}{n}\right)$

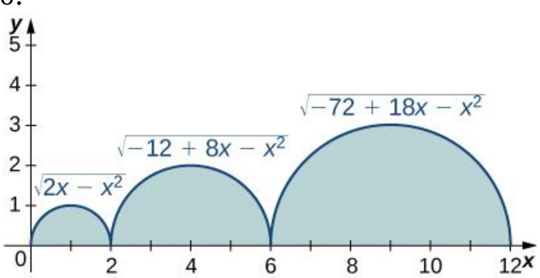
Answer: $\int_0^{2\pi} x\cos(x)\,dx$

69. $R_n = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\log\left(\left(1+\frac{i}{n}\right)^2\right)$

Answer: $\int_1^2 x\log\left(x^2\right)dx$

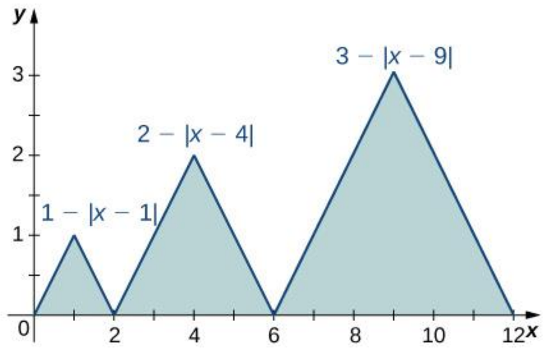
In the following exercises, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the *x*-axis.

70.



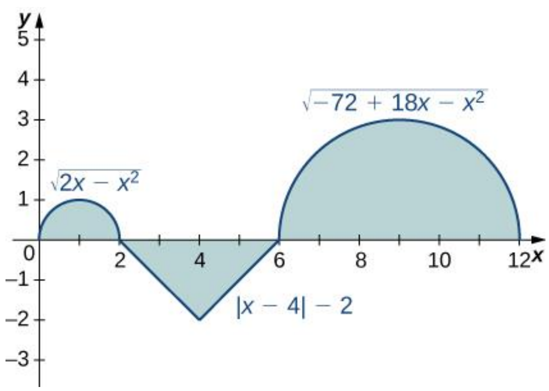
Answer: $\frac{\pi(1+4+9)}{2} = 7\pi$

71.



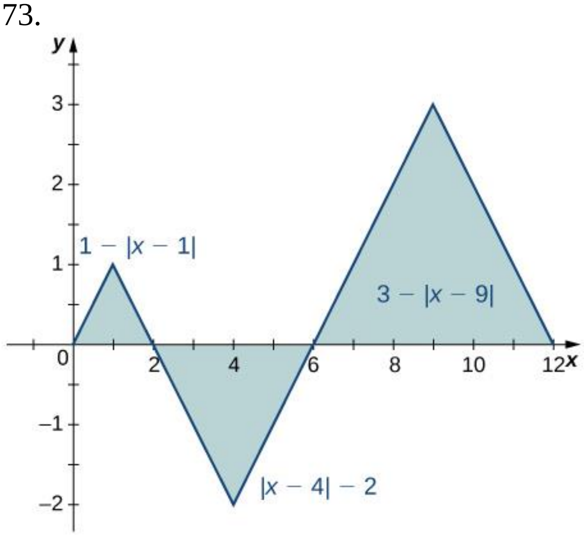
Answer: $1+2\cdot 2+3\cdot 3=14$

72.



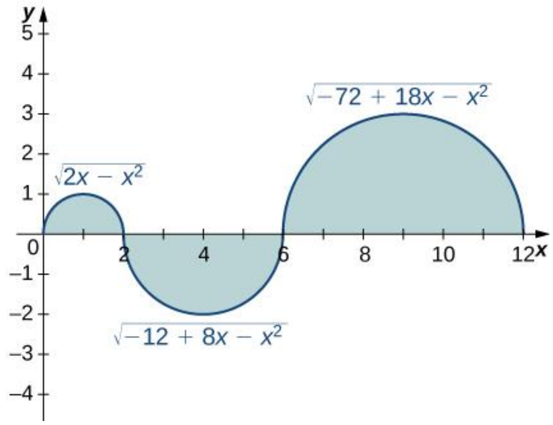
Answer: $\dfrac{\pi}{2}-4+\dfrac{9\pi}{2}-4=5\pi-4$

73.



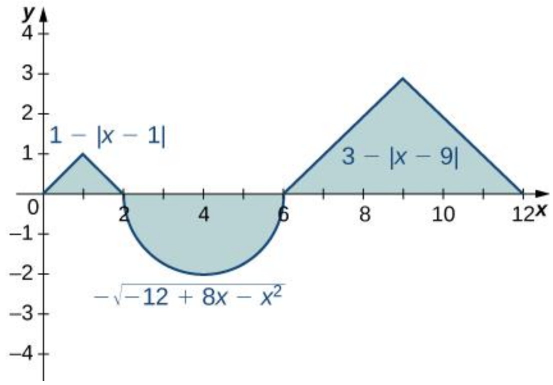
Answer: $1-4+9=6$

74.



Answer: $\dfrac{(1-4+9)\pi}{2}=3\pi$

75.



Answer: $1-2\pi+9=10-2\pi$

In the following exercises, evaluate the integral using area formulas.

76. $\int_0^3 (3-x)\,dx$

Answer: The integral is the area under the graph, $\dfrac{9}{2}$.

77. $\int_2^3 (3-x)\,dx$

Answer: The integral is the area of the triangle, $\dfrac{1}{2}$

78. $\int_{-3}^3 (3-|x|)\,dx$

Answer: The integral is the area of the triangle, 9.

79. $\int_0^6 (3-|x-3|)\,dx$

Answer: The integral is the area of the triangle, 9.

80. $\int_{-2}^2 \sqrt{4-x^2}\,dx$

Answer: The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

81. $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

Answer: The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

82. $\int_0^{12} \sqrt{36-(x-6)^2}\,dx$

Answer: The integral is the area $\frac{1}{2}\pi r^2 = 18\pi$.

83. $\int_{-2}^3 (3-|x|)\,dx$

Answer: The integral is the area of the “big” triangle less the “missing” triangle, $9-\frac{1}{2}$.

In the following exercises, use averages of values at the left (*L*) and right (*R*) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

84. $\{(0,0),(2,1),(4,3),(5,0),(6,0),(8,3)\}$ over $[0,8]$

Answer: $L = 0+2+3+0+0 = 5$, $R = 2+6+0+0+6 = 14$, $\frac{L+R}{2} = 8.5$

85. $\{(0,2),(1,0),(3,5),(5,5),(6,2),(8,0)\}$ over $[0,\ 8]$

Answer: $L = 2+0+10+5+4 = 21$, $R = 0+10+10+2+0 = 22$, $\frac{L+R}{2} = 21.5$

86. $\{(-4,-4),(-2,0),(0,-2),(3,3),(4,3)\}$ over $[-4,4]$

Answer: $L = -8+0-6+3 = -11$, $R = 0-4+9+3 = 8$, $\frac{L+R}{2} = -\frac{3}{2}$

87. $\{(-4,0),(-2,2),(0,0),(1,2),(3,2),(4,0)\}$ over $[-4,4]$

Answer: $L = 0+4+0+4+2 = 10$, $R = 4+0+2+4+0 = 10$, $\frac{L+R}{2} = 10$

Suppose that $\int_0^4 f(x)\,dx = 5$ and $\int_0^2 f(x)\,dx = -3,$ and $\int_0^4 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 2$. In the following exercises, compute the integrals.

88. $\int_0^4 \left(f(x)+g(x)\right)dx$

Answer: $\int_0^4 f(x)\,dx + \int_0^4 g(x)\,dx = 5-1=4$

89. $\int_2^4 \left(f(x)+g(x)\right)dx$

Answer: $\int_2^4 f(x)\,dx + \int_2^4 g(x)\,dx = 8-3=5$

90. $\int_0^2 \left(f(x)-g(x)\right)dx$

Answer: $\int_0^2 f(x)\,dx - \int_0^2 g(x)\,dx = -3-2=-5$

91. $\int_2^4 \left(f(x)-g(x)\right)dx$

Answer: $\int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx = 8+3=11$

92. $\int_0^2 \left(3f(x)-4g(x)\right)dx$

Answer: $3\int_0^2 f(x)\,dx - 4\int_0^2 g(x)\,dx = -9-8=-17$

93. $\int_2^4 \left(4f(x)-3g(x)\right)dx$

Answer: $4\int_2^4 f(x)\,dx - 3\int_2^4 g(x)\,dx = 32+9=41$

In the following exercises, use the identity $\int_{-A}^{A} f(x)\,dx = \int_{-A}^{0} f(x)\,dx + \int_0^A f(x)\,dx$ to compute the integrals.

94. $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2}\,dt$ (*Hint*: $\sin(-t) = -\sin(t)$)

Answer: The integrand is odd; the integral is zero.

95. $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t}\,dt$

Answer: The integrand is odd; the integral is zero.

96. $\int_1^3 (2-x)\,dx$ (*Hint:* Look at the graph of *f*.)

Answer: The signed area is zero because the graph produces two congruent triangles, one above the *x*-axis and the other below it.

97. $\int_2^4 (x-3)^3\,dx$ (*Hint:* Look at the graph of *f*.)

Answer: The integrand is antisymmetric with respect to $x=3$. The integral is zero.

In the following exercises, given that $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^1 x^2\,dx = \frac{1}{3}$, and $\int_0^1 x^3\,dx = \frac{1}{4}$, compute the integrals.

98. $\int_0^1 \left(1+x+x^2+x^3\right)dx$

Answer: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$

99. $\int_0^1 \left(1-x+x^2-x^3\right)dx$

Answer: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}$

100. $\int_0^1 (1-x)^2\,dx$

Answer: $\int_0^1 \left(1-2x+x^2\right)dx = 1-1+\frac{1}{3}=\frac{1}{3}$

101. $\int_0^1 (1-2x)^3\,dx$

Answer: $\int_0^1 \left(1-2x+4x^2-8x^3\right)dx = 1-1+\frac{4}{3}-2=-\frac{2}{3}$

102. $\int_0^1 \left(6x-\frac{4}{3}x^2\right)dx$

Answer: $6\cdot\frac{1}{2}-\frac{4}{3}\cdot\frac{1}{3}=\frac{23}{9}$

103. $\int_0^1 \left(7-5x^3\right)dx$

Answer: $7-\frac{5}{4}=\frac{23}{4}$

In the following exercises, use the comparison theorem

104. Show that $\int_0^3 \left(x^2-6x+9\right)dx \geq 0$.

Answer: The integrand is $\left(x-3\right)^2 \geq 0$.

105. Show that $\int_{-2}^3 \left(x-3\right)\left(x+2\right)dx \leq 0$.

Answer: The integrand is negative over $\left[-2,3\right]$.

106. Show that $\int_0^1 \sqrt{1+x^3}dx \leq \int_0^1 \sqrt{1+x^2}dx$.

Answer: $x^3 \leq x^2$ over $\left[0,1\right]$, so $\sqrt{1+x^3} \leq \sqrt{1+x^2}$ over $\left[0,1\right]$.

107. Show that $\int_1^2 \sqrt{1+x}\,dx \leq \int_1^2 \sqrt{1+x^2}dx$.

Answer: $x \leq x^2$ over $\left[1,2\right]$, so $\sqrt{1+x} \leq \sqrt{1+x^2}$ over $\left[1,2\right]$.

108. Show that $\int_0^{\pi/2} \sin t dt \geq \frac{\pi}{4}$. (*Hint:* $\sin t \geq \frac{2t}{\pi}$ over $\left[0,\frac{\pi}{2}\right]$)

Answer: Using the comparison theorem and $\frac{2}{\pi}\int_0^{\pi/2} t\,dt = \frac{\pi}{4}$ gives the estimate.

109. Show that $\int_{-\pi/4}^{\pi/4} \cos t dt \geq \pi\sqrt{2}/4$.

Answer: $\cos\left(t\right) \geq \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

In the following exercises, find the average value f_{ave} of f between a and b, and find a point c, where $f\left(c\right) = f_{\text{ave}}$.

110. $f\left(x\right) = x^2,\ a = -1,\ b = 1$

Answer: $f_{\text{ave}} = \frac{1}{3};\ c = \pm\frac{1}{\sqrt{3}}$

111. $f\left(x\right) = x^5,\ a = -1,\ b = 1$

Answer: $f_{\text{ave}} = 0;\ c = 0$

112. $f(x)=\sqrt{4-x^2}$, $a=0,\ b=2$

Answer: $\frac{\pi}{2}$ when $c=\pm\sqrt{4-\frac{\pi^2}{4}}$

113. $f(x)=(3-|x|),\ a=-3,\ b=3$

Answer: $\frac{3}{2}$ when $c=\pm\frac{3}{2}$

114. $f(x)=\sin x$, $a=0,\ b=2\pi$

Answer: $f_{\text{ave}}=0$; $c=0,\pi$

115. $f(x)=\cos x$, $a=0,\ b=2\pi$

Answer: $f_{\text{ave}}=0$; $c=\frac{\pi}{2},\frac{3\pi}{2}$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

116. **[T]** $y=\ln(x)$ over the interval $[1,\ 4]$; the exact solution is $\frac{\ln(256)}{3}-1$.

Answer: $\frac{L_{100}}{3}=0.8414$, $\frac{R_{100}}{3}=0.8553$; the exact average is between these values.

117. **[T]** $y=e^{x/2}$ over the interval $[0,\ 1]$; the exact solution is $2\left(\sqrt{e}-1\right)$.

Answer: $L_{100}=1.294$, $R_{100}=1.301$; the exact average is between these values.

118. **[T]** $y=\tan x$ over the interval $\left[0,\ \frac{\pi}{4}\right]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

Answer: $L_{100}\times\frac{4}{\pi}=0.4363$, $R_{100}\times\frac{4}{\pi}=0.4463$; the exact average is between these values.

119. **[T]** $y=\frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1,\ 1]$; the exact solution is $\frac{\pi}{6}$.

Answer: $L_{100}\times\left(\frac{1}{2}\right)=0.5178$, $R_{100}\times\left(\frac{1}{2}\right)=0.5294$

In the following exercises, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

120. **[T]** $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

Answer: $L_1 \times \left(\frac{1}{2}\right) = -2$, $L_{10} \times \left(\frac{1}{2}\right) = -2.86$, $L_{100} \times \left(\frac{1}{2}\right) = -2.687$. L_{100} is accurate in the first decimal digit.

121. **[T]** $y = x e^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}\left(e^4 - 1\right)$.

Answer: $L_1 = 0$, $L_{10} \times \left(\frac{1}{2}\right) = 8.743493$, $L_{100} \times \left(\frac{1}{2}\right) = 12.861728$. The exact answer ≈ 26.799, so L_{100} is not accurate.

122. **[T]** $y = \left(\frac{1}{2}\right)^x$ over the interval $[0,\ 4]$; the exact solution is $\frac{15}{64 \ln(2)}$.

Answer: $L_1 \times \left(\frac{1}{4}\right) = 1$, $L_{10} \times \left(\frac{1}{4}\right) = 0.3872$, $L_{100} \times \left(\frac{1}{4}\right) = 0.3428$. The exact answer ≈ 0.338, so L_{100} is not accurate to first decimal.

123. **[T]** $y = x \sin\left(x^2\right)$ over the interval $[-\pi,\ 0]$; the exact solution is $\frac{\cos\left(\pi^2\right) - 1}{2\pi}$.

Answer: $L_1 \times \left(\frac{1}{\pi}\right) = 1.352$, $L_{10} \times \left(\frac{1}{\pi}\right) = -0.1837$, $L_{100} \times \left(\frac{1}{\pi}\right) = -0.2956$. The exact answer ≈ -0.303, so L_{100} is not accurate to first decimal.

124. Suppose that $A = \int_0^{2\pi} \sin^2 t\,dt$ and $B = \int_0^{2\pi} \cos^2 t\,dt$. Show that $A + B = 2\pi$ and $A = B$.

Answer: Use $\sin^2\theta + 1 = \cos^2\theta = 1$. $A = B$ follows because cosine is a shift of sine and the functions are 2π-periodic.

125. Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = \pi$ and $B = \int_{-\pi/4}^{\pi/4} \tan^2 t\,dt$. Show that $A - B = \frac{\pi}{2}$.

Answer: Use $\tan^2\theta + 1 = \sec^2\theta$. Then, $B - A = \int_{-\pi/4}^{\pi/4} 1\,dx = \frac{\pi}{2}$.

126. Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to 1/2 Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to 1/2.

Answer: $\int_0^{2\pi} \sin^2 t dt = \pi,$ so divide by the length 2π of the interval. $\sin^2 t$ has period π, so yes, it is true.

127. Show that the average value of $\cos^2 t$ over $[0,\ 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$

Answer: $\int_0^{2\pi} \cos^2 t dt = \pi,$ so divide by the length 2π of the interval. $\cos^2 t$ has period π, so yes, it is true.

128. Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)dt > \int_a^b \ell(t)dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t)dt$ whenever $a \le c < d \le b$.

Answer: $p(x) - \ell(x)$ is quadratic, so it can have two roots at most. If $\int_a^b p(t)dt > \int_a^b \ell(t)dt,$ then $p(x) > \ell(x)$ for some, and hence all, $x \in [a,\ b]$. The result follows from the comparison theorem.

129. Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward ($a < 0$) and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A,\ B]$ is $\int_A^B (ax^2 + bx + c)dx$ as large as possible?

Answer: The integral is maximized when one uses the largest interval on which *p* is nonnegative. Thus, $A = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

130. Suppose $[a,\ b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \ge 0$ over $[a_{i-1},\ a_i]$ or $f \le 0$ over $[a_{i-1},\ a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)dt$.

a. Explain why $\int_a^b f(t)dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$.

Answer: a. This is an extension of the identity $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$ when $a < c < b$. b. Then, the integral inequality follows from $A_1 + \cdots + A_N \le |A_1| + \cdots + |A_N|$.

131. Suppose *f* and *g* are continuous functions such that $\int_c^d f(t)\,dt \le \int_c^d g(t)\,dt$ for every subinterval $[c,\ d]$ of $[a,\ b]$. Explain why $f(x) \le g(x)$ for all values of *x*.

Answer: If $f(t_0) > g(t_0)$ for some $t_0 \in [a,\ b]$, then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c,\ d]$, and then $\int_d^d f(t)\,dt > \int_c^d g(t)\,dt$ over this interval.

132. Suppose the average value of *f* over $[a,\ b]$ is 1 and the average value of *f* over $[b,\ c]$ is 1 where $a < c < b$. Show that the average value of *f* over $[a,\ c]$ is also 1.

Answer: The integral of *f* over an interval is the same as the integral of the average of *f* over that interval. Thus, $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^c 1dt + \int_c^b 1dt = (c-a)+(b-c) = b-a$. Dividing through by $b-a$ then gives the desired identity.

133. Suppose that $[a,\ b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of *f* over each subinterval $[a_{i-1},\ a_i] = 1$ is equal to 1 for each $i = 1, \cdots, N$. Explain why the average value of *f* over $[a,\ b]$ is also equal to 1.

Answer: The integral of *f* over an interval is the same as the integral of the average of *f* over that interval. Thus,

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_{a_0}^{a_1} f(t)\,dt + \int_{a_1}^{a_2} f(t)\,dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)\,dt = \int_{a_0}^{a_1} 1dt + \int_{a_1}^{a_2} 1dt + \cdots + \int_{a_{N+1}}^{a_N} 1dt \\ &= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a.\end{aligned}$$

Dividing through by $b-a$ gives the desired identity.

134. Suppose that for each *i* such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i$. Show that

$$\int_0^N f(t)\,dt = \frac{N(N+1)}{2}.$$

Answer: $\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

135. Suppose that for each *i* such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)\,dt = i^2$. Show that

$$\int_0^N f(t)\,dt = \frac{N(N+1)(2N+1)}{6}.$$

Answer: $\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

136. **[T]** Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t)=t^2$ over $[0,1]$. Given that $\int_0^1 t^2 dt = 0.\overline{33}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

Answer: $L_{10}=0.285$, $R_{10}=0.385$, $\frac{L_{10}+R_{10}}{2}=0.335,$ so the estimate is accurate to two decimal places.

137. **[T]** Compute the left and right Riemann sums, L_{10} and R_{10}, and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t)=\left(4-t^2\right)$ over $[1,2]$. Given that $\int_1^2 \left(4-t^2\right) dt = 1.\overline{66}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

Answer: $L_{10}=1.815$, $R_{10}=1.515$, $\frac{L_{10}+R_{10}}{2}=1.665,$ so the estimate is accurate to two decimal places.

138. If $\int_1^5 \sqrt{1+t^4}\,dt = 41.7133...$, what is $\int_1^5 \sqrt{1+u^4}\,du$?

Answer: Since we only changed the name of the integration variable, the value of the integral is the same, $41.7133\ldots$.

139. Estimate $\int_0^1 t\,dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t\,dt$?

Answer: The average is $1/2$, which is equal to the integral in this case.

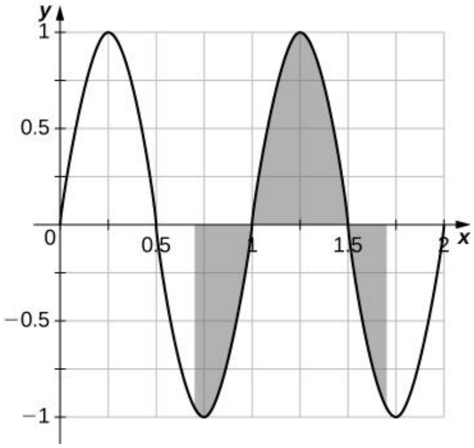
140. Estimate $\int_0^1 t\,dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t=\frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t\,dt$?

Answer: The midpoint estimate is $1/2$, which is equal to the integral.

141. From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t)\,dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)\,dt = 0$ for any value of *a*.



Answer: a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0,\ 1]$, so the average value is zero. b. For any value of *a*, the graph between $[a,\ a+1]$ is a shift of the graph over $[0,\ 1]$, so the net areas above and below the axis do not change and the average remains zero.

142. If *f* is 1-periodic ($f(t+1) = f(t)$), odd, and integrable over $[0,\ 1]$, is it always true that $\int_0^1 f(t)\,dt = 0$?

Answer: Yes, $\int_{-1/2}^{1/2} f = 0$. Since *f* is 1-periodic, the integral over any interval of length 1 is the same, hence equal to zero.

143. If *f* is 1-periodic and $\int_0^1 f(t)\,dt = A$, is it necessarily true that $\int_a^{1+a} f(t)\,dt = A$ for all *A*?
Answer: Yes, the integral over any interval of length 1 is the same.