Chapter 4
Applications of Derivatives
4.2 Linear Approximations and Differentials

Section Exercises

46. What is the linear approximation for any generic linear function $y = mx + b$?

Answer: $L(x) = mx + b$

47. Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

Answer: $f'(a) = 0$

48. Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

Answer: Answers will vary

49. When is the linear approximation exact?

Answer: The linear approximation exact when $y = f(x)$ is linear or constant.

For the following exercises, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

50. $f(x) = x + x^4,\ a = 0$

Answer: $L(x) = x$

51. $f(x) = \frac{1}{x},\ a = 2$

Answer: $L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$

52. $f(x) = \tan x,\ a = \frac{\pi}{4}$

Answer: $L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$

53. $f(x) = \sin x,\ a = \frac{\pi}{2}$

Answer: $L(x) = 1$

54. $f(x) = x \sin x,\ a = 2\pi$

Answer: $L(x) = 2\pi(x - 2\pi)$

55. $f(x)=\sin^2 x,\ a=0$
Answer: $L(x)=0$

For the following exercises, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and $a,$ and evaluating $L(x)=f(a)+f'(a)(x-a).$ Check your answer using a calculator.

56. **[T]** $(2.001)^6$
Answer: 64.192

57. **[T]** $\sin(0.02)$
Answer: 0.02

58. **[T]** $\cos(0.03)$
Answer: 1

59. **[T]** $(15.99)^{1/4}$
Answer: 1.9996875

60. **[T]** $\dfrac{1}{0.98}$
Answer: 1.02

61. **[T]** $\sin(3.14)$
Answer: 0.001593

For the following exercises, determine the appropriate $f(x)$ and $a,$ and evaluate $L(x)=f(a)+f'(a)(x-a).$ Calculate the numerical error in the linear approximations that follow.

62. **[T]** $(1.01)^3$
Answer: $1.03;$ error, ~ 0.0003

63. **[T]** $\cos(0.01)$
Answer: $1;$ error, ~ 0.00005

64. **[T]** $(\sin(0.01))^2$
Answer: $0;$ error, ~ 0.0001

65. **[T]** $(1.01)^{-3}$

Answer: 0.97; error, ~ 0.0006

66. **[T]** $\left(1+\frac{1}{10}\right)^{10}$

Answer: 2 ; error, ~ 0.6

67. **[T]** $\sqrt{8.99}$

Answer: $3-\frac{1}{600}$; error, $\sim 4.632\times 10^{-7}$

For the following exercises, find the differential of the function.

68. $y = 3x^4 + x^2 - 2x + 1$

Answer: $dy = \left(12x^3 + 2x - 2\right)dx$

69. $y = x\cos x$

Answer: $dy = \left(\cos x - x\sin x\right)dx$

70. $y = \sqrt{1+x}$

Answer: $dy = \left(\frac{1}{2\sqrt{x+1}}\right)dx$

71. $y = \frac{x^2+2}{x-1}$

Answer: $dy = \left(\frac{x^2-2x-2}{(x-1)^2}\right)dx$

For the following exercises, find the differential and evaluate for the given x and dx.

72. $y = 3x^2 - x + 6,\ \ x = 2,\ \ dx = 0.1$

Answer: $dy = \left(6x-1\right)dx,\ \ 1.1$

73. $y = \frac{1}{x+1},\ \ x = 1,\ \ dx = 0.25$

Answer: $dy = -\frac{1}{\left(x+1\right)^2}dx,\ \ -\frac{1}{16}$

74. $y = \tan x, \quad x = 0, \quad dx = \dfrac{\pi}{10}$

Answer: $dy = \sec^2(x)dx, \quad \dfrac{\pi}{10}$

75. $y = \dfrac{3x^2+2}{\sqrt{x+1}}, \quad x = 0, \quad dx = 0.1$

Answer: $dy = \dfrac{9x^2+12x-2}{2(x+1)^{3/2}}dx, \quad -0.1$

76. $y = \dfrac{\sin(2x)}{x}, \quad x = \pi, \quad dx = 0.25$

Answer: $dy = \dfrac{2x\cos(2x)-\sin(2x)}{x^2}dx, \quad \dfrac{1}{2\pi}$

77. $y = x^3 + 2x + \dfrac{1}{x}, \quad x = 1, \quad dx = 0.05$

Answer: $dy = \left(3x^2 + 2 - \dfrac{1}{x^2}\right)dx, \quad 0.2$

For the following exercises, find the change in volume dV or in surface area dA.

78. dV if the sides of a cube change from 10 to 10.1.
Answer: 30

79. dA if the sides of a cube change from x to $x + dx$.
Answer: $12xdx$

80. dA if the radius of a sphere changes from r by dr.
Answer: $8\pi rdr$

81. dV if the radius of a sphere changes from r by dr.
Answer: $4\pi r^2dr$

82. dV if a circular cylinder with $r = 2$ changes height from 3 cm to 3.05cm.
Answer: 0.2π cm^3

83. dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9$cm.
Answer: -1.2π cm^3

For the following exercises, use differentials to estimate the maximum and relative error when computing the surface area or volume.

84. A spherical golf ball is measured to have a radius of $5\text{mm},$ with a possible measurement error of $0.1\text{ mm}.$ What is the possible change in volume?

Answer: $10\pi\text{ mm}^3$

85. A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Answer: -100 ft^3

86. An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

Answer: $\frac{4}{15}\pi\text{ in}^3$

For the following exercises, confirm the approximations by using the linear approximation at $x=0.$

87. $\sqrt{1-x}\approx 1-\frac{1}{2}x$

88. $\frac{1}{\sqrt{1-x^2}}\approx 1$

89. $\sqrt{c^2+x^2}\approx c$