Chapter 4
Applications of Derivatives
4.4 The Mean Value Theorem

Section Exercises

148. Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

Answer: One example is $f(x)=\begin{cases} \dfrac{1}{x^2} & -1 \le x < 0, 0 < x \le 1 \\ 0 & x = 0 \end{cases}$

149. Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

Answer: One example is $f(x) = |x| + 3,\ -2 \text{ £ } x \text{ £ } 2$

150. When are Rolle’s theorem and the Mean Value Theorem equivalent?

Answer: When $f(a) = f(b)$

151. If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

Answer: Yes, but the Mean Value Theorem still does not apply

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.’

152. $y = \sin(\pi x)$

Answer: $(-\infty, \infty)$

153. $y = \dfrac{1}{x^3}$

Answer: $(-\infty, 0), (0, \infty)$

154. $y = \sqrt{4 - x^2}$

Answer: $(-2, 2)$

155. $y = \sqrt{x^2 - 4}$

Answer: $(-\infty, -2), (2, \infty)$

156. $y = \ln(3x - 5)$

Answer: $\left(\frac{5}{3}, \infty\right)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$.

157. **[T]** $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Answer: 2 points

158. **[T]** $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

Answer: 2 points

159. **[T]** $y = x^2 \cos(\pi x)$ over $[-2, 2]$

Answer: 5 points

160. **[T]** $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

Answer: 5 points

For the following exercises, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2-0)$.

161. $f(x) = x^3$

Answer: $c = \frac{2\sqrt{3}}{3}$

162. $f(x) = \sin(\pi x)$

Answer: $c = \frac{1}{2}, \frac{3}{2}$

163. $f(x) = \cos(2\pi x)$

Answer: $c = \frac{1}{2}, 1, \frac{3}{2}$

164. $f(x) = 1 + x + x^2$

Answer: $c = 1$

165. $f(x) = (x-1)^{10}$

Answer: $c = 1$

166. $f\left(x\right)=\left(x-1\right)^{9}$

Answer: $c=1\pm\frac{1}{\sqrt[4]{3}}$

For the following exercises, show there is no c **such that** $f\left(1\right)-f\left(-1\right)=f'\left(c\right)\left(2\right)$. **Explain why the Mean Value Theorem does not apply over the interval** $[-1,1]$.

167. $f\left(x\right)=\left|x-\frac{1}{2}\right|$

Answer: Not differentiable

168. $f(x)=\frac{1}{x^{2}}$

Answer: Not continuous

169. $f(x)=\sqrt{|x|}$

Answer: Not differentiable

170. $f\left(x\right)=\lfloor x\rfloor$ (*Hint*: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

Answer: Not continuous

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a,b]$. **Justify your answer.'**

171. $y=e^{x}$ over $[0,1]$

Answer: Yes

172. $y=\ln(2x+3)$ over $\left[-\frac{3}{2},0\right]$

Answer: Does not apply since $f(a)$ DNE

173. $f(x)=\tan(2\pi x)$ over $[0,2]$

Answer: The Mean Value Theorem does not apply since the function is discontinuous at $x=\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}$.

174. $y=\sqrt{9-x^{2}}$ over $[-3,3]$

Answer: Yes

175. $y = \dfrac{1}{|x+1|}$ over $[0,3]$

Answer: Yes

176. $y = x^3 + 2x + 1$ over $[0,6]$

Answer: Yes

177. $y = \dfrac{x^2 + 3x + 2}{x}$ over $[-1,1]$

Answer: The Mean Value Theorem does not apply; discontinuous at $x = 0$.

178. $y = \dfrac{x}{\sin(\pi x) + 1}$ over $[0,1]$

Answer: Yes

179. $y = \ln(x+1)$ over $[0, e-1]$

Answer: Yes

180. $y = x\sin(\pi x)$ over $[0,2]$

Answer: Yes

181. $y = 5 + |x|$ over $[-1,1]$

Answer: The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For the following exercises, consider the roots of the equation.

182. Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

Answer: $x = -4$

183. Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

Answer: $b = \pm 2\sqrt{c}$

184. Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

Answer: $b > 0,$ no

For the following exercises, use a calculator to graph the function over the interval $[a,b]$ and graph the secant line from a to $b.$ Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of $c,$ if possible, or write the final equation and use a calculator to estimate to four digits.

185. **[T]** $y=\tan(\pi x)$ over $\left[-\frac{1}{4},\frac{1}{4}\right]$

Answer: $c=\pm\frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right),\ c=\pm 0.1533$

186. **[T]** $y=\frac{1}{\sqrt{x+1}}$ over $[0,3]$

Answer: $c=3^{2/3}-1$

187. **[T]** $y=\left|x^2+2x-4\right|$ over $[-4,0]$

Answer: The Mean Value Theorem does not apply.

188. [**T**] $y=x+\frac{1}{x}$ over $\left[\frac{1}{2},4\right]$

Answer: $c=\sqrt{2}$

189. **[T]** $y=\sqrt{x+1}+\frac{1}{x^2}$ over $[3,8]$

Answer: $\frac{1}{2\sqrt{c+1}}-\frac{2}{c^3}=\frac{521}{2880};\ c=3.133,\ 5.867$

190. At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

Answer: Yes. Since your average speed was 65 mph, there was obviously somewhere within the 39 mi where you were driving more than 60 mph.

191. Two cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer: Yes

192. Show that $y=\sec^2 x$ and $y=\tan^2 x$ have the same derivative. What can you say about $y=\sec^2 x-\tan^2 x$?

Answer: It is constant.

193. Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer: It is constant.