Chapter 4
Applications of Derivatives
4.5 Derivatives and the Shape of a Graph

Section Exercises

194. If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

Answer: If f' does not change sign

195. For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Answer: It is not a local maximum/minimum because f' does not change sign

196. For the function $y = x^3$, is $x = 0$ an inflection point?

Answer: Yes

197. Is it possible for a point c to be both an inflection point and a local extrema of a twice differentiable function?

Answer: No

198. Why do you need continuity for the first derivative test? Come up with an example.

Answer: A piecewise function

199. Explain whether a concave-down function has to cross $y = 0$ for some value of x.

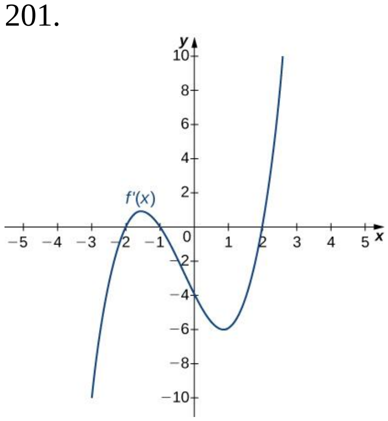
Answer: False; for example, $y = \sqrt{x}$.

200. Explain whether a polynomial of degree 2 can have an inflection point.

Answer: No, the second derivative is constant so it cannot change signs

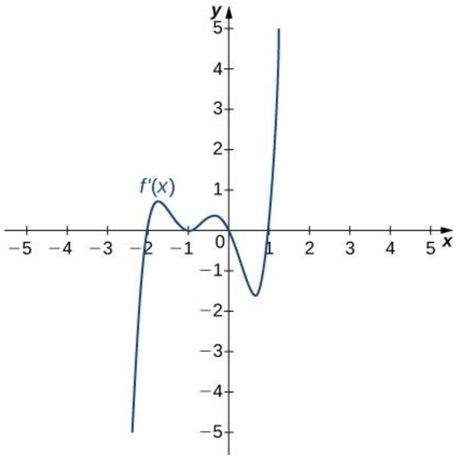
For the following exercises, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

201.



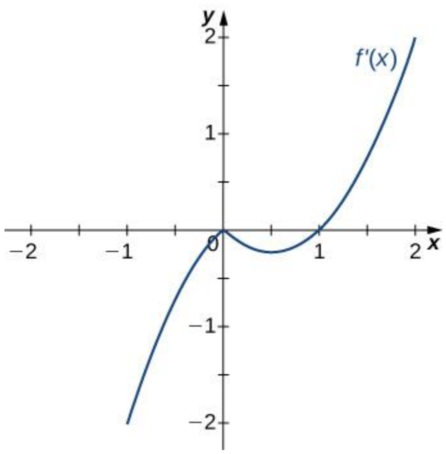
Answer: Increasing for $-2 < x < -1$ and $x > 2$; decreasing for $x < -2$ and $-1 < x < 2$

202.



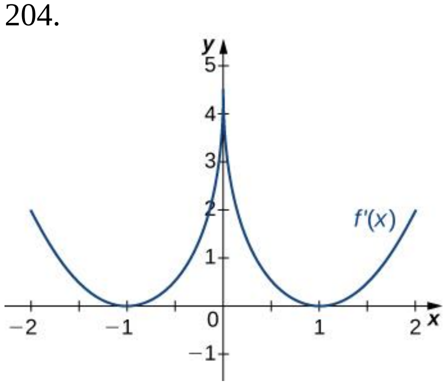
Answer: Decreasing for $x < -2, 0 < x < 1;$ increasing for $-2 < x < 0$ and $x > 1$

203.



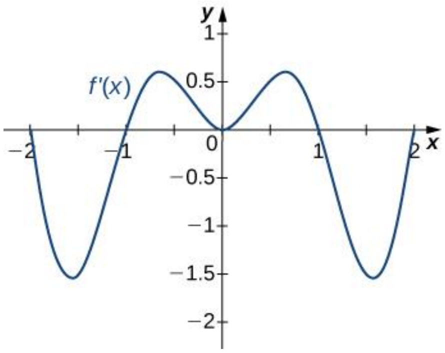
Answer: Decreasing for $x < 1,$ increasing for $x > 1$

204.



Answer: Increasing for all x

205.



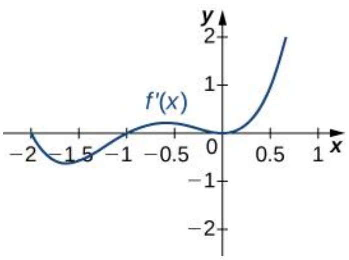
Answer: Decreasing for $-2 < x < -1$ and $1 < x < 2$; increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

For the following exercises, analyze the graphs of f', then list all intervals where

a. f is increasing and decreasing and

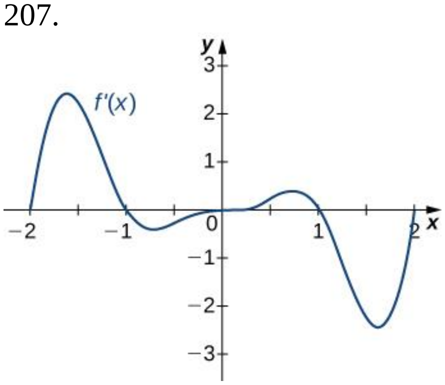
b. the minima and maxima are located.

206.



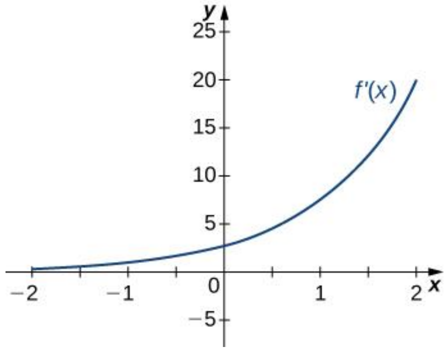
Answer: a. Increasing over $x > -1$, decreasing over $x < -1$; b. minimum at $x = -1$

207.



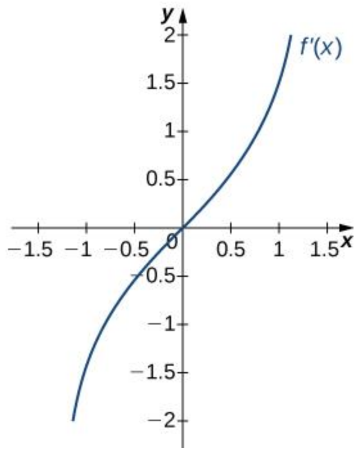
Answer: a. Increasing over $-2 < x < -1$, $0 < x < 1$, $x > 2$, decreasing over $x < -2$, $-1 < x < 0$, $1 < x < 2$; b. maxima at $x = -1$ and $x = 1$, minima at $x = -2$ and $x = 0$ and $x = 2$

208.



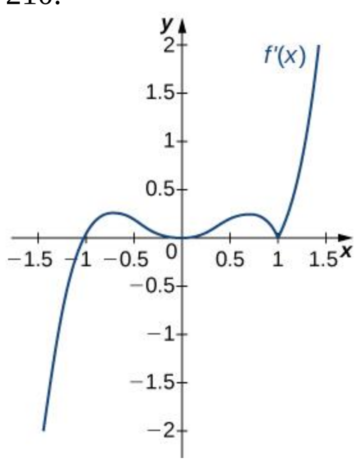
Answer: a. Increasing over $-2 < x < 2$, b. No local extrema

209.



Answer: a. Increasing over $x > 0$, decreasing over $x < 0$; b. Minimum at $x = 0$

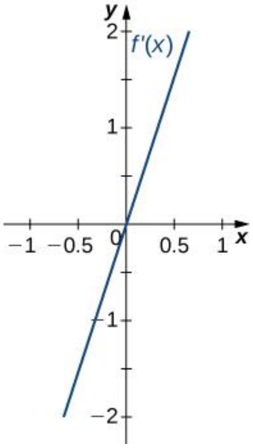
210.



Answer: a. Increasing over $x > -1$, decreasing over $x < -1$; b. Minimum at $x = -1$

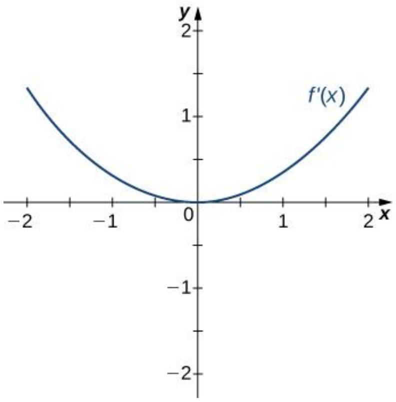
For the following exercises, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

211.



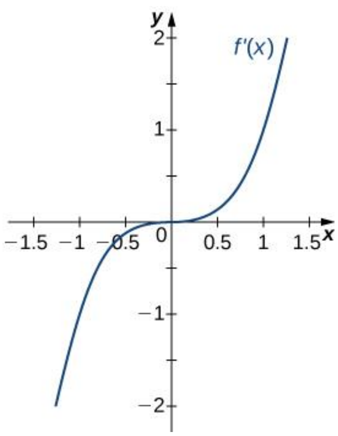
Answer: Concave up on all x, no inflection points

212.



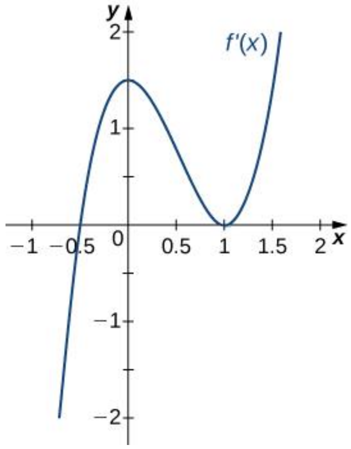
Answer: Concave up for $x > 0$, concave down for $x < 0$, inflection point at $x = 0$

213.



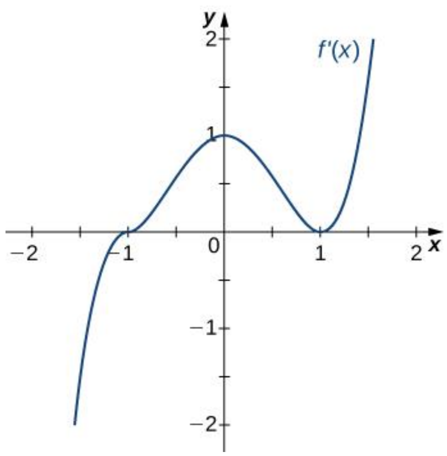
Answer: Concave up on all x, no inflection points

214.



Answer: Concave up for $x<0$ and $x>1,$ concave down for $0<x<1,$ inflection points at $x=0$ and $x=1$

215.



Answer: Concave up for $x<0$ and $x>1,$ concave down for $0<x<1,$ inflection points at $x=0$ and $x=1$

For the following exercises, draw a graph that satisfies the given specifications for the domain $x=[-3,3].$ The function does not have to be continuous or differentiable.

216. $f(x)>0,\ f'(x)>0$ over $x>1,\ -3<x<0,\ f'(x)=0$ over $0<x<1$

Answer: Answers will vary

217. $f'(x)>0$ over $x>2,\ -3<x<-1,\ f'(x)<0$ over $-1<x<2,\ f''(x)<0$ for all x

Answer: Answers will vary

218. $f''(x)<0$ over $-1<x<1,\ f''(x)>0, -3<x<-1,\ 1<x<3,$ local maximum at $x=0,$ local minima at $x=\pm 2$

Answer: Answers will vary

219. There is a local maximum at $x=2$, local minimum at $x=1$, and the graph is neither concave up nor concave down.

Answer: Answers will vary

220. There are local maxima at $x=\pm 1,$ the function is concave up for all $x,$ and the function remains positive for all $x.$

Answer: Answers will vary

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

221. $f(x)=\sin x+\sin^3 x$ over $-\pi < x < \pi$

Answer: a. Increasing over $-\frac{\pi}{2} < x < \frac{\pi}{2},$ decreasing over $x < -\frac{\pi}{2},\ x > \frac{\pi}{2}$ b. Local maximum at $x=\frac{\pi}{2};$ local minimum at $x=-\frac{\pi}{2}$

222. $f(x)=x^2+\cos x$

Answer: a. Increasing over $x>0,$ decreasing over $x<0$ b. Minimum at $x=0$

For the following exercises, determine a. intervals where f is concave up or concave down, and b. the inflection points of f.

223. $f(x)=x^3-4x^2+x+2$

Answer: a. Concave up for $x>\frac{4}{3}$, concave down for $x<\frac{4}{3}$ b. Inflection point at $x=\frac{4}{3}$

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. **the inflection points of f.**

224. $f(x)=x^2-6x$

Answer: a. Increasing over $x>3,$ decreasing over $x<3$ b. Minimum at $x=3$ c. Concave up over all x d. No inflection points

225. $f(x)=x^3-6x^2$

Answer: a. Increasing over $x<0$ and $x>4$, decreasing over $0<x<4$ b. Maximum at $x=0$, minimum at $x=4$ c. Concave up for $x>2,$ concave down for $x<2$ d. Infection point at $x=2$

226. $f(x) = x^4 - 6x^3$

Answer: a. Increasing over $x > 4.5$, decreasing over $x < 4.5$ b. Minimum at $x = 4.5$ c. Concave up for $x < 0$ and $x > 3$, concave down for $0 < x < 3$ d. Inflection points at $x = 0$, $x = 3$

227. $f(x) = x^{11} - 6x^{10}$

Answer: a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over $0 < x < \frac{60}{11}$ b. Minimum at $x = \frac{60}{11}$ c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$ d. Inflection point at $x = \frac{54}{11}$

228. $f(x) = x + x^2 - x^3$

Answer: a. Increasing for $-\frac{1}{3} < x < 1$, decreasing for $x > 1$, $x < -\frac{1}{3}$ b. Maximum at $x = 1$, minimum at $x = -\frac{1}{3}$ c. Concave up for $x < \frac{1}{3}$, concave down for $x > \frac{1}{3}$ d. Inflection point at $x = \frac{1}{3}$

229. $f(x) = x^2 + x + 1$

Answer: a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$ b. Minimum at $x = -\frac{1}{2}$ c. Concave up for all x d. No inflection points

230. $f(x) = x^3 + x^4$

Answer: a. Increasing over $x > -\frac{3}{4}$, decreasing over $x < -\frac{3}{4}$ b. Minimum at $x = -\frac{3}{4}$ c. Concave up for $x < -\frac{1}{2}$, $x > 0$; concave down for $-\frac{1}{2} < x < 0$ d. Inflection points at $x = -\frac{1}{2}$, $x = 0$

For the following exercises, determine

a. **intervals where f is increasing or decreasing,**
b. **local minima and maxima of f,**
c. **intervals where f is concave up and concave down, and**
d. **the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.**

231. **[T]** $f(x)=\sin(\pi x)-\cos(\pi x)$ over $x=[-1,1]$

Answer: a. Increases over $-\frac{1}{4}<x<\frac{3}{4}$, decreases over $x>\frac{3}{4}$ and $x<-\frac{1}{4}$ b. Minimum at $x=-\frac{1}{4}$, maximum at $x=\frac{3}{4}$ c. Concave up for $-\frac{3}{4}<x<\frac{1}{4}$, concave down for $x<-\frac{3}{4}$ and $x>\frac{1}{4}$ d. Inflection points at $x=-\frac{3}{4}$, $x=\frac{1}{4}$

232. **[T]** $f(x)=x+\sin(2x)$ over $x=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Answer: a. Increasing over $-\frac{\pi}{3}<x<\frac{\pi}{3}$, decreasing over $-\frac{\pi}{2}<x<-\frac{\pi}{3}$ and $\frac{\pi}{3}<x<\frac{\pi}{2}$ b. Minimum at $x=-\frac{\pi}{3}$, maximum at $x=\frac{\pi}{3}$ c. Concave up for $x<0$, concave down for $x>0$ d. Inflection point at $x=0$

233. **[T]** $f(x)=\sin x+\tan x$ over $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Answer: a. Increasing for all x b. No local minimum or maximum c. Concave up for $x>0$, concave down for $x<0$ d. Inflection point at $x=0$

234. **[T]** $f(x)=(x-2)^2(x-4)^2$

Answer: a. Increasing over $2<x<3$, $x>4$; decreasing over for $x<2$, $3<x<4$ b. Maximum at $x=3$, minima at $x=2$ and $x=4$ c. Concave down for $\frac{1}{3}\left(9-\sqrt{3}\right)<x<\frac{1}{3}\left(9+\sqrt{3}\right)$, concave up for $x<\frac{1}{3}\left(9-\sqrt{3}\right)$, $x>\frac{1}{3}\left(9+\sqrt{3}\right)$ d. Inflection points at $x=\frac{1}{3}\left(9-\sqrt{3}\right)$, $x=\frac{1}{3}\left(9+\sqrt{3}\right)$

235. **[T]** $f(x)=\frac{1}{1-x}$, $x\neq 1$

Answer: a. Increasing for all x where defined b. No local minima or maxima c. Concave up for $x<1$; concave down for $x>1$ d. No inflection points in domain

236. **[T]** $f(x)=\frac{\sin x}{x}$ over $x=[2\pi,0)\cup(0,2\pi]$

Answer: a. Increasing over $x>4.493, -4.493<x<0,$ decreasing over $0<x<4.493,\ \ x<-4.493$ b. No maximum, minima at $x=\pm 4.493$ c. Concave up for $2.082<x<5.940,\ \ -5.940<x<-2.082;$ concave down for $-2.082<x<2.082,\ x>5.940,\ x<-5.940$ d. Inflection points at $x=\pm 5.940,\ x=\pm 2.082$

237. $f(x)=\sin(x)e^x$ over $x=[-\pi,\pi]$

Answer: a. Increasing over $-\frac{\pi}{4}<x<\frac{3\pi}{4},$ decreasing over $x>\frac{3\pi}{4},\ x<-\frac{\pi}{4}$ b. Minimum at $x=-\frac{\pi}{4},$ maximum at $x=\frac{3\pi}{4}$ c. Concave up for $-\frac{\pi}{2}<x<\frac{\pi}{2},$ concave down for $x<-\frac{\pi}{2},\ x>\frac{\pi}{2}$ d. Infection points at $x=\pm\frac{\pi}{2}$

238. $f(x)=\ln x\sqrt{x}, x>0$

Answer: a. Increasing over $x>\frac{1}{e^2},$ decreasing over $0<x<\frac{1}{e^2}$ b. Minimum at $x=\frac{1}{e^2}$ c. Concave up for $0<x<1$, concave down for $x>1$ d. Inflection point at $x=1$

239. $f(x)=\frac{1}{4}\sqrt{x}+\frac{1}{x}, x>0$

Answer: a. Increasing over $x>4,$ decreasing over $0<x<4$ b. Minimum at $x=4$ c. Concave up for $0<x<8\sqrt[3]{2},$ concave down for $x>8\sqrt[3]{2}$ d. Inflection point at $x=8\sqrt[3]{2}$

240. $f(x)=\frac{e^x}{x}, x\neq 0$

Answer: a. Decreasing for $x<0$ and $0<x<1,$ increasing over $x>1$ b. Local minimum at $x=1$ c. Concave up for $x>0,$ concave down for $x<0$ d. No inflection points

For the following exercises, interpret the sentences in terms of f, f', and f''.

241. The population is growing more slowly. Here f is the population.

Answer: $f>0, f'>0, f''<0$

242. A bike accelerates faster, but a car goes faster. Here $f=$ Bike's position minus Car's position.

Answer: $f<0, f'<0, f''>0$

243. The airplane lands smoothly. Here f is the plane's altitude.

Answer: $f>0, f'<0, f''<0$

244. Stock prices are at their peak. Here f is the stock price.
Answer: $f > 0, f' = 0, f'' < 0$

245. The economy is picking up speed. Here f is a measure of the economy, such as GDP.
Answer: $f > 0, f' > 0, f'' > 0$

For the following exercises, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0, f'(3) = 0.$ Determine whether the following statements are true or false. Justify your answer.

246. $f(x) = 0$ for some $1 \le x \le 3$
Answer: False, imagine $f(1) = 1, f(3) = 3,$ and f increases for $1 < x < 3$

247. $f''(x) = 0$ for some $1 \le x \le 3$
Answer: True, by the Mean Value Theorem

248. There is no absolute maximum at $x = 3$
Answer: True, since $f(x)$ has an odd-degree leading term it has no absolute maximum.

249. If $f(x)$ has three roots, then it has 1 inflection point.
Answer: True, examine derivative

250. If $f(x)$ has one inflection point, then it has three real roots.
Answer: False, the function could have only one real root.