Chapter 4
Applications of Derivatives
4.9 Newton's Method

Section Exercises

For the following exercises, write Newton's formula as $x_{n+1} = F(x_n)$ **for solving** $f(x) = 0$

406. $f(x) = x^2 + 1$

Answer: $F(x_n) = x_n - \dfrac{x_n{}^2 + 1}{2x_n}$

407. $f(x) = x^3 + 2x + 1$

Answer: $F(x_n) = x_n - \dfrac{x_n{}^3 + 2x_n + 1}{3x_n{}^2 + 2}$

408. $f(x) = \sin x$

Answer: $F(x_n) = x_n + \dfrac{\sin(x_n)}{\cos(x_n)}$

409. $f(x) = e^x$

Answer: $F(x_n) = x_n - \dfrac{e^{x_n}}{e^{x_n}}$

410. $f(x) = x^3 + 3xe^x$

Answer: $F(x_n) = x_n - \dfrac{x^3{}_n + 3x_n e^{x_n}}{3x^2{}_n + (3x_n + 3)e^{x_n}}$

For the following exercises, solve $f(x) = 0$ **using the iteration** $x_{n+1} = x_n - cf(x_n)$, **which differs slightly from Newton's method. Find a** c **that works and a** c **that fails to converge, with the exception of** $c = 0$.

411. $f(x) = x^2 - 4$, with $x_0 = 0$

Answer: $|c| > 0.5$ fails, $|c| \le 0.5$ works

412. $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

Answer: $|c| > 1$ fails, $|c| \le 1$ works

413. What is the value of "c" for Newton's method?

Answer: $c = \frac{1}{f'(x_n)}$

For the following exercises, start at

a. $x_0 = 0.6$ **and**

b. $x_0 = 2.$

Compute x_1 **and** x_2 **using the specified iterative method**.

414. $x_{n+1} = x_n{}^2 - \frac{1}{2}$

Answer: a. $x_1 = -\frac{7}{50}, x_2 = -\frac{1201}{2500}$; b. $x_1 = \frac{7}{2}, x_2 = \frac{47}{4}$

415. $x_{n+1} = 2x_n(1 - x_n)$

Answer: a. $x_1 = \frac{12}{25}, x_2 = \frac{312}{625}$; b. $x_1 = -4, x_2 = -40$

416. $x_{n+1} = \sqrt{x_n}$

Answer: a. $x_1 = 0.7746, x_2 = 0.8801$; b. $x_1 = 1.414, x_2 = 1.189$

417. $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Answer: a. $x_1 = 1.291, x_2 = 0.8801$; b. $x_1 = 0.7071, x_2 = 1.189$

418. $x_{n+1} = 3x_n(1 - x_n)$

Answer: a. $x_1 = \frac{18}{25}, x_2 = \frac{378}{625}$; b. $x_1 = -6, x_2 = -126$

419. $x_{n+1} = x_n{}^2 + x_n - 2$

Answer: a. $x_1 = -\frac{26}{25}, x_2 = -\frac{1224}{625}$; b. $x_1 = 4, x_2 = 18$

420. $x_{n+1} = \frac{1}{2}x_n - 1$

Answer: a. $x_1 = -\frac{7}{10}, x_2 = -\frac{27}{20}$; b. $x_1 = 0, x_2 = -1$

421. $x_{n+1} = \left| x_n \right|$

Answer: a. $x_1 = \dfrac{6}{10}, x_2 = \dfrac{6}{10}$; b. $x_1 = 2, x_2 = 2$

For the following exercises, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

422. $x^2 - 10 = 0$
Answer: 3.1623 or -3.1623

423. $x^4 - 100 = 0$
Answer: 3.1623 or -3.1623

424. $x^2 - x = 0$
Answer: 0 or 1

425. $x^3 - x = 0$
Answer: $0, -1$ or 1

426. $x + 5\cos(x) = 0$
Answer: $-1.3064, 1.9774,$ or $3.8375)$

427. $x + \tan(x) = 0$, choose $x_0 \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right)$

Answer: 0

428. $\dfrac{1}{1-x} = 2$

Answer: $\dfrac{1}{2}$

429. $1 + x + x^2 + x^3 + x^4 = 2$
Answer: 0.5188 or -1.2906

430. $x^3 + (x+1)^3 = 10^3$
Answer: 7.4055

431. $x = \sin^2(x)$
Answer: 0

For the following exercises, use Newton's method to find the fixed points of the function where $f(x) = x;$ round to three decimals.

432. $\sin x$
Answer: $x = 0$

433. $\tan(x)$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
Answer: 4.493

434. $e^x - 2$
Answer: $-1.841, 1.146$

435. $\ln(x) + 2$
Answer: $0.159, 3.146$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. For the following exercises, consider the formulation of the method.

436. To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0.$ Show that to solve for the critical points of a function $f(x),$ Newton's method is given by $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$
Answer: $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$

437. What additional restrictions are necessary on the function f ?
Answer: We need f to be twice continuously differentiable.)

For the following exercises, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

438. Minimum of $f(x) = x^2 + 2x + 4$
Answer: $x = -1$

439. Minimum of $f(x) = 3x^3 + 2x^2 - 16$
Answer: $x = 0$

440. Minimum of $f(x) = x^2e^x$
Answer: $x = 0$)

441. Maximum of $f(x) = x + \frac{1}{x}$
Answer: $x = -1$

442. Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$
Answer: $x = -5.805$

443. Maximum of $f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$
Answer: $x = 5.619$

444. Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$
Answer: $x = -2.289$

445. Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$
Answer: $x = -1.326$

For the following exercises, use the specified method to solve the equation. If it does not work, explain why it does not work.

446. Newton's method, $x^2 + 2 = 0$
Answer: There are no real solutions to the equation.

447. Newton's method, $0 = e^x$
Answer: There is no solution to the equation.

448. Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$
Answer: x_1 is undefined; also, there are no real solutions to the equation.

449. Solving $x_{n+1} = -x_n{}^3$ starting at $x_0 = -1$
Answer: It enters a cycle.

For the following exercises, use the secant method, an alternative iterative method to Newton's method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

450. Find a root to $0 = x^2 - x - 3$ accurate to three decimal places.
Answer: -1.303 or 2.303

451. Find a root to $0 = \sin x + 3x$ accurate to four decimal places.
Answer: 0

452. Find a root to $0 = e^x - 2$ accurate to four decimal places.
Answer: 0.6932

453. Find a root to $\ln(x+2) = \frac{1}{2}$ accurate to four decimal places.
Answer: -0.3513

454. Why would you use the secant method over Newton's method? What are the necessary restrictions on f?
Answer: We do not need differentiability of f for the secant method.

For the following exercises, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

455. $f(x) = x^2 + 2x + 1, x_0 = 1$
Answer: Newton: 11 iterations, secant: 16 iterations

456. $f(x) = x^2, x_0 = 1$
Answer: Newton: 10 iterations, secant: 15 iterations

457. $f(x) = \sin x, x_0 = 1$
Answer: Newton: three iterations, secant: six iterations

458. $f(x) = e^x - 1, x_0 = 2$
Answer: Newton: six iterations, secant: eight iterations

459. $f(x) = x^3 + 2x + 4, x_0 = 0$
Answer: Newton: five iterations, secant: eight iterations

In the following exercises, consider Kepler's equation regarding planetary orbits, $M = E - \varepsilon \sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

460. Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon = 0.25;$ round to three decimals.

Answer: $E = 1.287$

461. Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon = 0.8;$ round to three decimals.

Answer: $E = 4.071$

The following two exercises consider a bank investment. The initial investment is $\$10,000.$ **After** 25 **years, the investment has tripled to** $\$30,000.$

462. Use Newton's method to determine the interest rate if the interest was compounded annually.

Answer: 4.492%

463. Use Newton's method to determine the interest rate if the interest was compounded continuously.

Answer: 4.394%

464. The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \left(\frac{1}{2}\right)x^{2/3}.$

Use Newton's method to find the break-even point if the printer sells each book for $\$20$.

Answer: 127 books

Student Project
Iterative Processes and Chaos

1. Let $r = 0.5$ and choose $x_0 = 0.2.$ Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, $2-\text{cycle},\ 4-\text{cycle}$.)?

Answer: Converges to 0

2. What happens when $r = 2$?

Answer: Converges to 0.5

3. For $r = 3.2$ and $r = 3.5,$ calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?

Answer: $r = 3.2$ yields a 2-cycle. $r = 3.5$ yields a 4-cycle.

4. Now let $r = 4.$ Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?

Answer: $r = 4$ gives chaotic behavior.

5. Repeat the process for $r = 4,$ but let $x_0 = 0.201.$ How does this behavior compare with the behavior for $x_0 = 0.2$?

Answer: Changing the initial condition to $x_0 = 0.201$ changes the sequence values fairly drastically. By the 9th or 10th iteration the values are more than 0.5 apart.