Chapter 4
Applications of Derivatives
4.10 Antiderivatives

Section Exercises

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

465. $F(x)=5x^3+2x^2+3x+1,\ f(x)=15x^2+4x+3$

Answer: $F'(x)=15x^2+4x+3$

466. $F(x)=x^2+4x+1,\ f(x)=2x+4$

Answer: $F'(x)=2x+4$

467. $F(x)=x^2e^x,\ f(x)=e^x\left(x^2+2x\right)$

Answer: $F'(x)=2xe^x+x^2e^x$

468. $F(x)=\cos x,\ f(x)=-\sin x$

Answer: $F'(x)=-\sin x$

469. $F(x)=e^x,\ f(x)=e^x$

Answer: $F'(x)=e^x$

For the following exercises, find the antiderivative of the function.

470. $f(x)=\dfrac{1}{x^2}+x$

Answer: $F(x)=\dfrac{x^2}{2}-\dfrac{1}{x}+C$

471. $f(x)=e^x-3x^2+\sin x$

Answer: $F(x)=e^x-x^3-\cos(x)+C$

472. $f(x)=e^x+3x-x^2$

Answer: $F(x)=e^x-\dfrac{x^3}{3}+\dfrac{3x^2}{2}+C$

473. $f(x) = x - 1 + 4\sin(2x)$

Answer: $F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

474. $f(x) = 5x^4 + 4x^5$

Answer: $F(x) = x^5 + \frac{2}{3}x^6 + C$

475. $f(x) = x + 12x^2$

Answer: $F(x) = \frac{1}{2}x^2 + 4x^3 + C$

476. $f(x) = \frac{1}{\sqrt{x}}$

Answer: $F(x) = 2\sqrt{x} + C$

477. $f(x) = \left(\sqrt{x}\right)^3$

Answer: $F(x) = \frac{2}{5}\left(\sqrt{x}\right)^5 + C$

478. $f(x) = x^{1/3} + (2x)^{1/3}$

Answer: $F(x) = \frac{3}{4}x^{4/3} + \frac{3\sqrt[3]{2}}{4}x^{4/3} + C$

479. $f(x) = \frac{x^{1/3}}{x^{2/3}}$

Answer: $F(x) = \frac{3}{2}x^{2/3} + C$

480. $f(x) = 2\sin(x) + \sin(2x)$

Answer: $F(x) = -2\cos(x) - \frac{1}{2}\cos(2x) + C$

481. $f(x) = \sec^2(x) + 1$

Answer: $F(x) = x + \tan(x) + C$

482. $f(x) = \sin x \cos x$

Answer: $F(x) = -\frac{1}{2}\cos^2(x) + C$

483. $f(x) = \sin^2(x)\cos(x)$

Answer: $F(x) = \frac{1}{3}\sin^3(x) + C$

484. $f(x) = 0$

Answer: $F(x) = C$

485. $f(x) = \frac{1}{2}\csc^2(x) + \frac{1}{x^2}$

Answer: $F(x) = -\frac{1}{2}\cot(x) - \frac{1}{x} + C$

486. $f(x) = \csc x \cot x + 3x$

Answer: $F(x) = \frac{3}{2}x^2 - \csc x + C$

487. $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer: $F(x) = -\sec x - 4\csc x + C$

488. $f(x) = 8\sec x(\sec x - 4\tan x)$

Answer: $F(x) = 8\tan x - 32\sec x + C$

489. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

Answer: $F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$

For the following exercises, evaluate the integral.

490. $\int(-1)\,dx$

Answer: $-x + C$

491. $\int \sin x dx$

Answer: $-\cos x + C$

492. $\int \left(4x + \sqrt{x}\right) dx$

Answer: $2x^2 + \frac{2}{3}x^{3/2} + C$

493. $\int \frac{3x^2 + 2}{x^2} dx$

Answer: $3x - \frac{2}{x} + C$

494. $\int (\sec x \tan x + 4x) dx$

Answer: $2x^2 + \sec x + C$

495. $\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx$

Answer: $\frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$

496. $\int \left(x^{-1/3} - x^{2/3}\right) dx$

Answer: $\frac{3}{2}x^{2/3} - \frac{3}{5}x^{5/3} + C$

497. $\int \frac{14x^3 + 2x + 1}{x^3} dx$

Answer: $14x - \frac{2}{x} - \frac{1}{2x^2} + C$

498. $\int \left(e^x + e^{-x}\right) dx$

Answer: $e^x - e^{-x} + C$

For the following exercises, solve the initial value problem.

499. $f'(x) = x^{-3}$, $f(1) = 1$

Answer: $f(x) = -\frac{1}{2x^2} + \frac{3}{2}$

500. $f'(x)=\sqrt{x}+x^2,\ f(0)=2$

Answer: $f(x)=\frac{2}{3}\sqrt{x^3}+\frac{1}{3}x^3+2$

501. $f'(x)=\cos x+\sec^2(x),\ f\left(\frac{\pi}{4}\right)=2+\frac{\sqrt{2}}{2}$

Answer: $f(x)=\sin x+\tan x+1$

502. $f'(x)=x^3-8x^2+16x+1,\ f(0)=0$

Answer: $f(x)=\frac{1}{4}x^4-\frac{8}{3}x^3+8x^2+x$

503. $f'(x)=\frac{2}{x^2}-\frac{x^2}{2},\ f(1)=0$

Answer: $f(x)=-\frac{1}{6}x^3-\frac{2}{x}+\frac{13}{6}$

For the following exercises, find two possible functions f given the second- or third-order derivatives.

504. $f''(x)=x^2+2$

Answer: Answers may vary; one possible answer is $f(x)=\frac{1}{12}x^4+x^2$

505. $f''(x)=e^{-x}$

Answer: Answers may vary; one possible answer is $f(x)=e^{-x}$

506. $f''(x)=1+x$

Answer: : Answers may vary; one possible answer is $f(x)=\frac{x^3}{6}+\frac{x^2}{2}$

507. $f'''(x)=\cos x$

Answer: Answers may vary; one possible answer is $f(x)=-\sin x$

508. $f'''(x)=8e^{-2x}-\sin x$

Answer: Answers may vary; one possible answer is $f(x)=-e^{-2x}-\cos x$

509. A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec^2. How long before the car stops?

Answer: 5.867 sec

510. In the preceding problem, calculate how far the car travels in the time it takes to stop.

Answer: 172.1 ft

511. You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec^2. How long does it take you to reach merging speed at 60 mph?

Answer: 7.333 sec

512. Based on the previous problem, how far does the car travel to reach merging speed?

Answer: 322.7 ft

513. A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer: 13.75 ft/sec^2

514. A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer: 8.604 ft/sec^2

For the following exercises, find the antiderivative of the function, assuming $F(0)=0$.

515. **[T]** $f(x)=x^2+2$

Answer: $F(x)=\frac{1}{3}x^3+2x$

516. **[T]** $f(x)=4x-\sqrt{x}$

Answer: $F(x)=2x^2-\frac{2}{3}\sqrt{x^3}$

517. **[T]** $f(x)=\sin x+2x$

Answer: $F(x)=x^2-\cos x+1$

518. **[T]** $f(x)=e^x$

Answer: $F(x)=e^x-1$

519. **[T]** $f(x)=\frac{1}{(x+1)^2}$

Answer: $F(x)=-\frac{1}{(x+1)}+1$

520. **[T]** $f(x)=e^{-2x}+3x^2$

Answer: $F(x)=-\frac{1}{2}e^{-2x}+x^3+\frac{1}{2}$

For the following exercises, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

521. If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer: True

522. If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

Answer: False

523. If $f(x)$ is the antiderivative of $v(x)$, then $f(x)+1$ is the antiderivative of $v(x)+1$.

Answer: False

524. If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

Answer: False

Chapter Review Exercises

***True or False*? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.**

525. If $f(-1)=-6$ and $f(1)=2,$ then there exists at least one point $x\in[-1,1]$ such that $f'(x)=4.$

Answer: True, by Mean Value Theorem

526. If $f'(c)=0,$ there is a maximum or minimum at $x=c.$

Answer: False, for example $f(x)=x^3$ at $x=0$

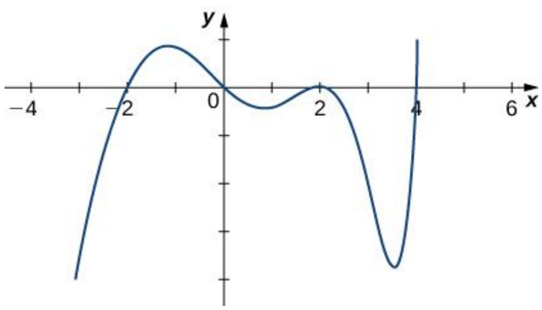
527. There is a function such that $f(x)<0,$ $f'(x)>0,$ and $f''(x)<0.$ (A graphical "proof" is acceptable for this answer.)

Answer: True

528. There is a function such that there is both an inflection point and a critical point for some value $x=a.$

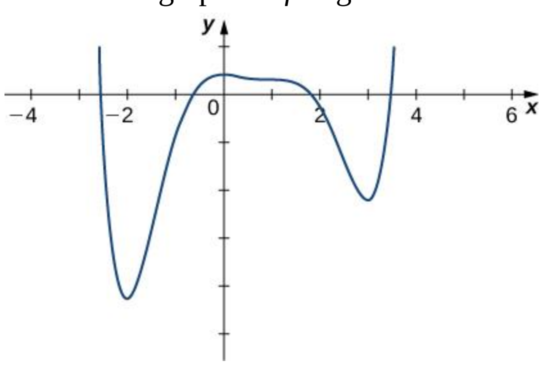
Answer: True; example, $f(x)=x^3$ at $x=0$

529. Given the graph of $f',$ determine where f is increasing or decreasing.

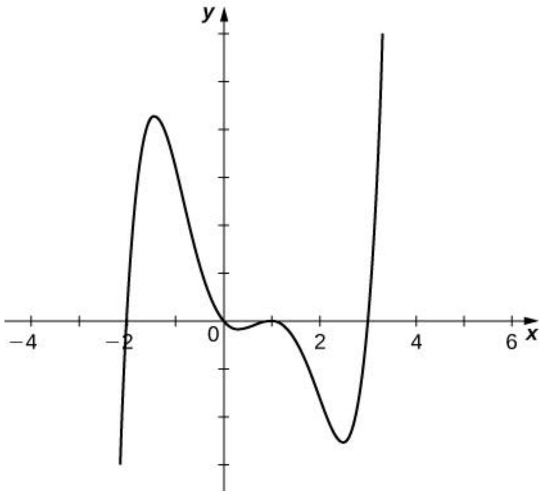


Answer: Increasing: $(-2,0)\cup(4,\infty),$ decreasing: $(-\infty,-2)\cup(0,4)$

530. The graph of f is given below. Draw $f'.$



Answer:



531. Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Answer: $L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$

532. Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Answer: $dy = (2x - 5)\,dx, -0.1$

Find the critical points and the local and absolute extrema of the following functions on the given interval.

533. $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Answer: Critical point: $x = \frac{3\pi}{4}$, absolute minimum: $x = 0$, absolute maximum: $x = \pi$

534. $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Answer: Critical points: $x = -1, 0, 2$, absolute minimum: $x = 2$; absolute maximum: $x = -3$; local minimum: $x = -1$; local maximum: $x = 0$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

535. $x(t) = 3t^4 - 8t^3 - 18t^2$

Answer: Increasing: $(-1, 0) \cup (3, \infty)$, decreasing: $(-\infty, -1) \cup (0, 3)$, concave up: $\left(-\infty, \frac{1}{3}\left(2 - \sqrt{13}\right)\right) \cup \left(\frac{1}{3}\left(2 + \sqrt{13}\right), \infty\right)$, concave down: $\left(\frac{1}{3}\left(2 - \sqrt{13}\right), \frac{1}{3}\left(2 + \sqrt{13}\right)\right)$

536. $y = x + \sin(\pi x)$

Answer: Increasing: $(2k - 0.6031, 2k + 0.6031)$, decreasing: $(2k + 0.6031, 2k - 0.6031)$, concave up: $(2k + 1, 2k)$, concave down: $(2k, 2k + 1)$ for each integer k

537. $g(x) = x - \sqrt{x}$

Answer: Increasing: $\left(\frac{1}{4}, \infty\right)$, decreasing: $\left(0, \frac{1}{4}\right)$, concave up: $(0, \infty)$, concave down: nowhere

538. $f(\theta) = \sin(3\theta)$

Answer: Increasing: $\left(\frac{\pi}{6}(4k - 1), \frac{\pi}{6}(4k + 1)\right)$, decreasing: $\left(\frac{\pi}{6}(4k + 1), \frac{\pi}{6}(4k + 3)\right)$, concave up: $\left(\frac{\pi}{3}(2k + 1), \frac{2k\pi}{3}\right)$, concave down: $\left(\frac{2k\pi}{3}, \frac{\pi}{3}(2k + 1)\right)$ for each integer k

Evaluate the following limits.

539. $\lim_{x\to\infty} \frac{3x\sqrt{x^2 + 1}}{\sqrt{x^4 - 1}}$

Answer: 3

540. $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

Answer: 1

541. $\lim_{x\to 1} \frac{x - 1}{\sin(\pi x)}$

Answer: $-\frac{1}{\pi}$

542. $\lim_{x\to\infty} (3x)^{1/x}$

Answer: 1

Use Newton's method to find the first two iterations, given the starting point.

543. $y = x^3 + 1, x_0 = 0.5$

Answer: $x_1 = -1, x_2 = -1$

544. $\frac{1}{x+1} = \frac{1}{2}, x_0 = 0$

Answer: $x_1 = \frac{1}{2}, x_2 = \frac{7}{8}$

Find the antiderivatives $F(x)$ of the following functions.

545. $g(x) = \sqrt{x} - \frac{1}{x^2}$

Answer: $F(x) = \frac{2x^{3/2}}{3} + \frac{1}{x} + C$

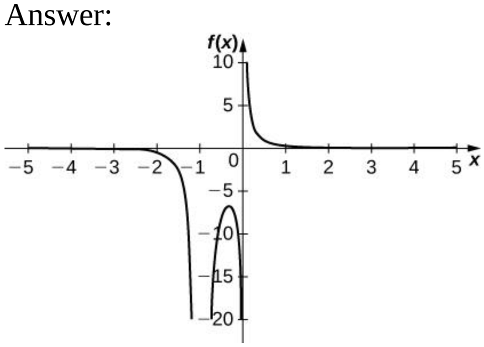
546. $f(x) = 2x + 6\cos x, F(\pi) = \pi^2 + 2$

Answer: $F(x) = x^2 + 6\sin x + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

547. $y = \frac{1}{x(x+1)^2}$

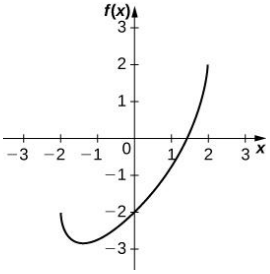
Answer:



Inflection points: none; critical points: $x = -\frac{1}{3}$; zeros: none; vertical asymptotes: $x = -1,\ x = 0$; horizontal asymptote: $y = 0$

548. $y = x - \sqrt{4 - x^2}$

Answer:



Inflection points: none, critical point: $x = -\sqrt{2}$, zeros: $x = \sqrt{2}$, asymptotes: none

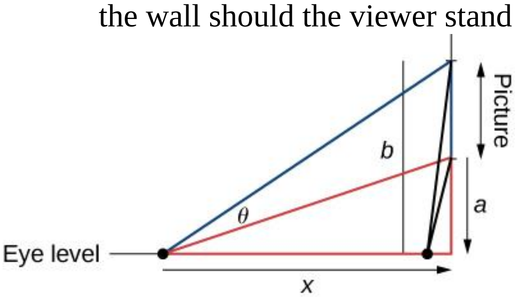
549. A car is being compacted into a rectangular solid. The volume is decreasing at a rate of 2 m^3/sec. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

Answer: The height is decreasing at a rate of 0.125 m/sec

550. A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right) m(t) v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

Answer: The kinetic energy is increasing at a rate of 25 MJ/sec.

551. The famous Regiomontanus' problem for angle maximization was proposed during the 15 th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



Answer: $x = \sqrt{ab}$ feet

552. An airline sells tickets from Tokyo to Detroit for $\$1200$. There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

Answer: Fare: $\$950$, passengers: 475