Chapter 3
Derivatives
3.9 Derivatives of Exponential and Logarithmic Functions

Section Exercises

For the following exercises, find $f'(x)$ for each function.

331. $f(x)=x^2e^x$

Answer: $2xe^x+x^2e^x$

332. $f(x)=\dfrac{e^{-x}}{x}$

Answer: $\dfrac{-xe^{-x}-e^{-x}}{x^2}$

333. $f(x)=e^{x^3\ln x}$

Answer: $e^{x^3\ln x}\left(3x^2\ln x+x^2\right)$

334. $f(x)=\sqrt{e^{2x}+2x}$

Answer: $\dfrac{1}{2}\left(e^{2x}+2x\right)^{-1/2}\left(2e^{2x}+2\right)$

335. $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

Answer: $\dfrac{4}{\left(e^x+e^{-x}\right)^2}$

336. $f(x)=\dfrac{10^x}{\ln 10}$

Answer: 10^x

337. $f(x)=2^{4x}+4x^2$

Answer: $2^{4x+2}\cdot\ln 2+8x$

338. $f(x)=3^{\sin 3\text{x}}$

Answer: $3^{\sin 3\text{x}+1}\cdot\ln 3\cdot\cos 3x$

339. $f(x)=x^{\pi}\cdot\pi^x$

Answer: $\pi x^{\pi-1}\cdot\pi^x+x^{\pi}\cdot\pi^x\ln\pi$

340. $f(x) = \ln\left(4x^3 + x\right)$

Answer: $\dfrac{12x^2 + 1}{4x^3 + x}$

341. $f(x) = \ln\sqrt{5x - 7}$

Answer: $\dfrac{5}{2(5x - 7)}$

342. $f(x) = x^2 \ln 9x$

Answer: $2x \ln 9x + x$

343. $f(x) = \log(\sec x)$

Answer: $\dfrac{\tan x}{\ln 10}$

344. $f(x) = \log_7\left(6x^4 + 3\right)^5$

Answer: $\dfrac{120x^3}{\ln 7\left(6x^4 + 3\right)}$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

Answer: $2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \dfrac{2x \ln 7}{\ln 3}$

For the following exercises, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

Answer: $x^{\sqrt{x}} \cdot \left(\dfrac{\ln x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right)$

347. $y = (\sin 2x)^{4x}$

Answer: $(\sin 2x)^{4x}\left[4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x\right]$

348. $y = (\ln x)^{\ln x}$

Answer: $(\ln x)^{\ln x} \cdot \left[\dfrac{\ln(\ln x) + 1}{x}\right]$

349. $y = x^{\log_2 x}$

Answer: $x^{\log_2 x} \cdot \dfrac{2\ln x}{x\ln 2}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

Answer: $\left(x^2 - 1\right)^{\ln x} \cdot \left[\dfrac{\ln\left(x^2 - 1\right)}{x} + \dfrac{2x\ln x}{x^2 - 1}\right]$

351. $y = x^{\cot x}$

Answer: $x^{\cot x} \cdot \left[-csc^2 x \cdot \ln x + \dfrac{\cot x}{x}\right]$

352. $y = \dfrac{x + 11}{\sqrt[3]{x^2 - 4}}$

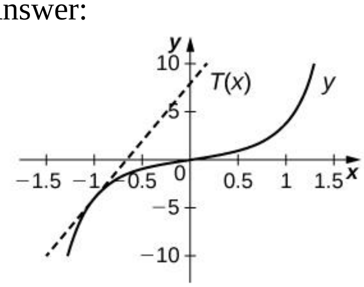
Answer: $\dfrac{x + 11}{\sqrt[3]{x^2 - 4}} \cdot \left[\dfrac{1}{x + 11} - \dfrac{2x}{3\left(x^2 - 4\right)}\right]$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

Answer: $x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4 \cdot \left[\dfrac{-1}{2x} + \dfrac{4x}{3\left(x^2 + 3\right)} + \dfrac{12}{3x - 4}\right]$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where $x = -1$. Graph both the function and the tangent line.

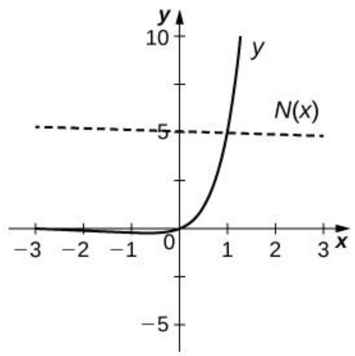
Answer:



$y = 12x + 8$

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

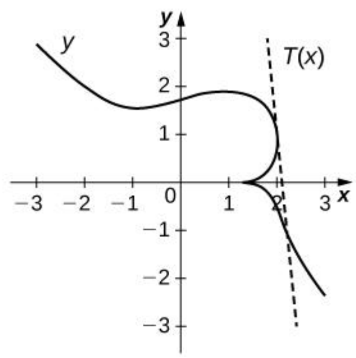
Answer:



$$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$$

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

Answer:



$y = 21 - 10x$

357. Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.

b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Answer: a. $x = e \sim 2.718$ b. $(e, \infty), (0, e)$

358. The formula $I(t)=\frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

Answer: a. (i) 0, (ii) $e^{-\pi/2}\sim 0.2079,$ (iii) 0, (iv) $e^{-3\pi/2}\sim -0.009,$ (v) 0, (vi) $e^{-5\pi/2}\sim 0.0004,$ (vii) 0, (viii) $e^{-7\pi/2}\sim -0.00002,$ (ix) 0 b. $I(t)$ $=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}.$

359. **[T]** The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use a. to determine the rate at which the population is increasing in t years.

c. Use b. to determine the rate at which the population is increasing in 10 years.

Answer: a. $P=500,000(1.05)^t$ individuals b. $P'(t)=24395\cdot(1.05)^t$ individuals per year c. $39,737$ individuals per year

360. **[T]** An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t=4$ hours.

Answer: a. $Q=9\cdot\left(\frac{1}{2}\right)^{t/12}$ b. $Q'(t)=\frac{3}{4}\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{t/12}$ c. -0.4126 g/hr

361. **[T]** The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function $N(t)=5.3e^{0.093t^2-0.87t}$, $(0\le t\le 4),$ where

$N(t)$ gives the number of cases (in thousands) and *t* is measured in years, with $t=0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

Answer: a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1964 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

362. **[T]** The *relative rate of change* of a differentiable function $y=f(x)$ is given by $\frac{100\cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x)=ae^{-b\cdot e^{-cx}}$ where a, b , and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.
b. Use a. to find the relative rate of change of a population in $x=20$ months when $a=204,\ b=0.0198,$ and $c=0.15.$
c. Briefly interpret what the result of b. means.

Answer: a. $100\cdot(bc)\cdot e^{-cx}$ b. 0.01479 c. In 20 months the relative rate of change is approximately 0.015% per month.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.
New York City Population Over Time

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

363. **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer: $p = 35741(1.045)^t$

364. **[T]** Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

Answer:

Years since 1790	P'
0	1573
10	2443
20	3794
30	5892
40	9150
50	14210
60	22068
70	34271

365. **[T]** Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer:

Years since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

366. **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from a.?

Answer: a. No, because population growth is increasing and accelerating; the model will grow too fast. b. 573.8 million. It overestimates, as expected

Chapter Review Exercises
***True or False*. Justify the answer with a proof or a counterexample.**

367. Every function has a derivative.
Answer: False.

368. A continuous function has a continuous derivative.
Answer: False

369. A continuous function has a derivative.
Answer: False

370. If a function is differentiable, it is continuous.
Answer: True

Use the limit definition of the derivative to exactly evaluate the derivative.

371. $f(x)=\sqrt{x+4}$
Answer: $\frac{1}{2\sqrt{x+4}}$

372. $f(x)=\frac{3}{x}$
Answer: $-\frac{3}{x^2}$

Find the derivatives of the following functions.

373. $f(x)=3x^3-\frac{4}{x^2}$
Answer: $9x^2+\frac{8}{x^3}$

374. $f(x)=\left(4-x^2\right)^3$
Answer: $-6x\left(4-x^2\right)^2$

375. $f(x)=e^{\sin x}$
Answer: $e^{\sin x}\cos x$

376. $f(x) = \ln(x+2)$

Answer: $\frac{1}{x+2}$

377. $f(x) = x^2 \cos x + x \tan(x)$

Answer: $x\sec^2(x) + 2x\cos(x) + \tan(x) - x^2 \sin(x)$

378. $f(x) = \sqrt{3x^2 + 2}$

Answer: $\frac{3x}{\sqrt{3x^2+2}}$

379. $f(x) = \frac{x}{4}\sin^{-1}(x)$

Answer: $\frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)\right)$

380. $x^2 y = (y+2) + xy\sin(x)$

Answer: $-\frac{y(\sin x + x(\cos x - 2))}{1 + x\sin x - x^2}$

Find the following derivatives of various orders.

381. First derivative of $y = x\ln(x)\cos x$

Answer: $\cos x \cdot (\ln x + 1) - x\ln(x)\sin x$

382. Third derivative of $y = (3x+2)^2$

Answer: 0

383. Second derivative of $y = 4^x + x^2 \sin(x)$

Answer: $4^x(\ln 4)^2 + 2\sin x + 4x\cos x - x^2 \sin x$

Find the equation of the tangent line to the following equations at the specified point.

384. $y = \cos^{-1}(x) + x$ at $x = 0$

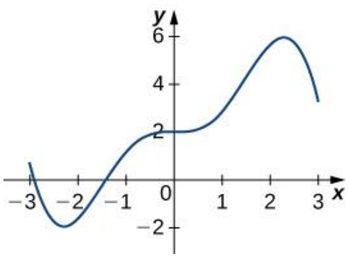
Answer: $T = \frac{\pi}{2}$

385. $y = x + e^x - \frac{1}{x}$ at $x = 1$

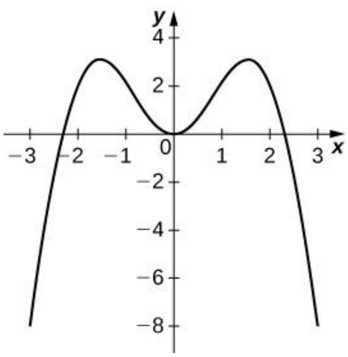
Answer: $T = (2 + e)x - 2$

Draw the derivative for the following graphs.

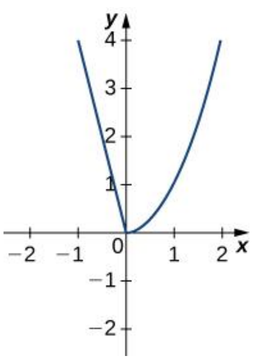
386.



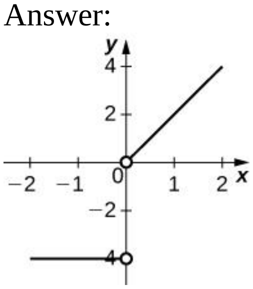
Answer:



387.



Answer:



The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t)=1.9+2.9\cos\left(\frac{\pi}{6}t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

388. Find and graph the derivative. What is the physical meaning?

Answer: $w'(t)=-\frac{2.9\pi}{6}\sin\left(\frac{\pi}{6}t\right)$. This is the rate at which the tide is changing.

389. Find $w'(3)$. What is the physical meaning of this value?

Answer: $w'(3)=-\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.
Wind Speeds of Hurricane Katrina

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

390. Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

Answer: 1.5. The wind speed is increasing at a rate of 1.5 mph/hr.

391. Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer: -7.5. The wind speed is decreasing at a rate of 7.5 mph/hr