Chapter 3
Derivatives
3.6 The Chain Rule

Section Exercises

For the following exercises, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

214. $y = 3u - 6,\ u = 2x^2$

Answer: $3 \cdot 4x = 12x$

215. $y = 6u^3,\ u = 7x - 4$

Answer: $18u^2 \cdot 7\ = 18(7x-4)^2 \cdot 7$

216. $y = \sin u,\ u = 5x - 1$

Answer: $\cos u \cdot 5 = \cos(5x-1) \cdot 5$

217. $y = \cos u,\ u = \frac{-x}{8}$

Answer: $-\sin u \cdot \frac{-1}{8}\ = -\sin\left(\frac{-x}{8}\right) \cdot \frac{-1}{8}$

218. $y = \tan u,\ u = 9x + 2$

Answer: $sec^2 u \cdot 9 =\ sec^2(9x+2) \cdot 9$

219. $y = \sqrt{4u+3}, u = x^2 - 6x$

Answer: $\frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$

For each of the following exercises,

- **a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and**
- **b. find $\frac{dy}{dx}$ as a function of x.**

220. $y = (3x-2)^6$

Answer: a. $f(u) = u^6, u = 3x - 2$; b. $18 \cdot (3x-2)^5$

221. $y=\left(3x^2+1\right)^3$

Answer: a. $u=3x^2+1$; b. $18x\left(3x^2+1\right)^2$

222. $y=\sin^5\left(x\right)$

Answer: a. $u=\sin x$; b. $5\sin^4\left(x\right)\cos\left(x\right)$

223. $y=\left(\frac{x}{7}+\frac{7}{x}\right)^7$

Answer: a. $f\left(u\right)=u^7, u=\frac{x}{7}+\frac{7}{x}$; b. $7\left(\frac{x}{7}+\frac{7}{x}\right)^6\cdot\left(\frac{1}{7}-\frac{7}{x^2}\right)$

224. $y=\tan\left(\sec x\right)$

Answer: a. $f\left(u\right)=\tan u,\ u=\sec x$; b. $\sec^2(\sec x)\cdot\sec x\tan x$

225. $y=\csc\left(\pi x+1\right)$

Answer: a. $f\left(u\right)=\csc u,\ u=\pi x+1$; b. $-\pi\csc\left(\pi x+1\right)\cdot\cot\left(\pi x+1\right)$

226. $y=cot^2 x$

Answer: a. $f\left(u\right)=u^2,\ u=\cot x$; a. $-2\cot x\cdot\csc^2 x$

227. $y=-6(\sin x)^{-3}$

Answer: a. $f\left(u\right)=-6u^{-3}, u=\sin x$, b. $18(\sin x)^{-4}\cdot\cos x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

228. $y=\left(3x^2+3x-1\right)^4$

Answer: $4\left(3x^2+3x-1\right)^3\cdot\left(6x+3\right)$

229. $y=\left(5-2x\right)^{-2}$

Answer: $\frac{4}{\left(5-2x\right)^3}$

230. $y=\cos^3\left(\pi x\right)$

Answer: $-3\pi\cos^2\left(\pi x\right)\sin\left(\pi x\right)$

231. $y=\left(2x^3-x^2+6x+1\right)^3$

Answer: $6\left(2x^3-x^2+6x+1\right)^2\left(3x^2-x+3\right)$

232. $y=\dfrac{1}{\sin^2(x)}$

Answer: $-2\cot(x)\csc^2(x)$

233. $y=\left(\tan x+\sin x\right)^{-3}$

Answer: $-3\left(\tan x+\sin x\right)^{-4}\cdot\left(\sec^2 x+\cos x\right)$

234. $y=x^2 cos^4 x$

Answer: $2x\cdot\cos^4 x-4x^2\cos^3 x\sin x$

235. $y=\sin\left(\cos 7x\right)$

Answer: $-7\cos\left(\cos 7x\right)\cdot\sin 7x$

236. $y=\sqrt{6+\sec\pi x^2}$

Answer: $\pi x\cdot\left(6+\sec\pi x^2\right)^{-1/2}\cdot\sec\pi x^2\tan\pi x^2$

237. $y=\cot^3\left(4x+1\right)$

Answer: $-12\cot^2(4x+1)\cdot\csc^2(4x+1)$

238. Let $y=[f(x)]^3$ and suppose that $f'(1)=4$ and $\dfrac{dy}{dx}=10$ for $x=1$. Find $f(1)$.

Answer: $\sqrt{\dfrac{5}{6}}$

239. Let $y=\left(f(x)+5x^2\right)^4$ and suppose that $f(-1)=-4$ and $\dfrac{dy}{dx}=3$ when $x=-1$. Find $f'(-1)$

Answer: $10\dfrac{3}{4}$

240. Let $y=\left(f\left(u\right)+3x\right)^2$ and $u=x^3-2x$. If $f\left(4\right)=6$ and $\frac{dy}{dx}=18$ when $x=2$, find $f'\left(4\right)$.

Answer: $\frac{-9}{40}$

241. **[T]** Find the equation of the tangent line to $y=-\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer: $y=\frac{-1}{2}x]$

242. **[T]** Find the equation of the tangent line to $y=\left(3x+\frac{1}{x}\right)^2$ at the point $\left(1,16\right)$. Use a calculator to graph the function and the tangent line together.

Answer: $y=16x$

243. Find the x-coordinates at which the tangent line to $y=\left(x-\frac{6}{x}\right)^8$ is horizontal.

Answer: $x=\pm\sqrt{6}$

244. **[T]** Find an equation of the line that is normal to $g(\theta)=\sin^2\left(\pi\theta\right)$ at the point $\left(\frac{1}{4},\frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

Answer: $y=-\frac{1}{\pi}\theta+\frac{1}{2}+\frac{1}{4\pi}$

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	-2	3	0
2	4	4	1	-1
3	3	-3	2	3

245. $h\left(x\right)=f\left(g\left(x\right)\right);\ a=0$

Answer: 10

246. $h(x)=g\left(f(x)\right);\ a=0$

Answer: -5

247. $h(x)=\left(x^4+g(x)\right)^{-2};\ a=1$

Answer: $-\frac{1}{8}$

248. $h(x)=\left(\frac{f(x)}{g(x)}\right)^2;\ a=3$

Answer: $-\frac{45}{4}$

249. $h(x)=f\left(x+f(x)\right);\ a=1$

Answer: -4

250. $h(x)=\left(1+g(x)\right)^3;\ a=2$

Answer: -12

251. $h(x)=g\left(2+f\left(x^2\right)\right);\ a=1$

Answer: -12

252. $h(x)=f\left(g(\sin x)\right);\ a=0$

Answer: 10

253. **[T]** The position function of a freight train is given by $s(t)=100(t+1)^{-2}$, with s in meters and t in seconds. At time $t=6$ s, find the train's

a. velocity and
b. acceleration.
c. Using a. and b. is the train speeding up or slowing down?

Answer: a. $-\frac{200}{343}$ m/s, b. $\frac{600}{2401}$ m/s^2, c. The train is slowing down since velocity and acceleration have opposite signs.

254. **[T]** A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches: $s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right)$.

   a. Determine the position of the spring at $t = 1.5$ s.
   b. Find the velocity of the spring at $t = 1.5$ s.

Answer: a. –2.12 in. b. -6.664 in./s^2

255. **[T]** The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

   a. Find the marginal cost $C'(x)$.
   b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.
   c. Use b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer: a. $C^{'}(x) = 0.0003x^2 - 0.04x + 3$ b. $\frac{dC}{dt} = 100 \cdot \left(0.0003x^2 - 0.04x + 3\right)$ c. Approximately $90,300 per week

256. **[T]** The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

   a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.
   b. Use a. to find the rate at which the area is expanding at $t = 4$ s.

Answer: a. $\frac{dA}{dt} = \frac{400\pi r}{(t+7)^3}$ b. 1.1080 in.^{2}/s

257. **[T]** The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

   a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.
   b. Use a. to find the rate at which the volume is changing at $t = 1$ min.

Answer: a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3}$ b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min.

258. **[T]** The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

Answer: $\sim -1.309°$ F/hr

259. **[T]** The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8,$ where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Answer: ~ 2.3 ft/hr