Chapter 2
Limits
2.4 Continuity

Section Exercises

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

131. $f(x)=\dfrac{1}{\sqrt{x}}$

Answer: The function is defined for all *x* in the interval $(0,\ \infty)$.

132. $f(x)=\dfrac{2}{x^2+1}$

Answer: The function is continuous for all real values *x*.

133. $f(x)=\dfrac{x}{x^2-x}$

Answer: Removable discontinuity at $x=0$; infinite discontinuity at $x=1$

134. $g(t)=t^{-1}+1$

Answer: Infinite discontinuity at $t=0$

135. $f(x)=\dfrac{5}{e^x-2}$

Answer: Infinite discontinuity at $x=\ln 2$

136. $f(x)=\dfrac{|x-2|}{x-2}$

Answer: Jump discontinuity at $x=2$

137. $H(x)=\tan 2x$

Answer: Infinite discontinuities at $x=\dfrac{(2k+1)\pi}{4}$, for $k=0,\pm1,\pm2,\pm3,\ldots$

138. $f(t)=\dfrac{t+3}{t^2+5t+6}$

Answer: Removable discontinuity at $t=-3$; infinite discontinuity at $t=-2$]

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

139. $f(x) = \dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

Answer: No. It is a removable discontinuity.

140. $h(\theta) = \dfrac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

Answer: No. It is an infinite discontinuity.

141. $g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1} & \text{if } u \neq \dfrac{1}{2} \\ \dfrac{7}{2} & \text{if } u = \dfrac{1}{2} \end{cases}$, at $u = \dfrac{1}{2}$

Answer: Yes. It is continuous.

142. $f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

Answer: No. It is a removable discontinuity.

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$

Answer: Yes. It is continuous.

144. $f(x) = \begin{cases} x\sin(x) & \text{if } x \leq \pi \\ x\tan(x) & \text{if } x > \pi \end{cases}$, at $x = \pi$

Answer: Yes. It is continuous.

In the following exercises, find the value(s) of *k* that makes each function continuous over the given interval.

145. $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$

Answer: $k = -5$

146. $f(\theta) = \begin{cases} \sin\theta, & 0 \leq \theta < \dfrac{\pi}{2} \\ \cos(\theta + k), & \dfrac{\pi}{2} \leq \theta \leq \pi \end{cases}$

Answer: $k=-\dfrac{\pi}{2}$

147. $f(x)=\begin{cases}\dfrac{x^2+3x+2}{x+2}, & x\neq -2\\ k, & x=-2\end{cases}$

Answer: $k=-1$

148. $f(x)=\begin{cases}e^{kx}, & 0\le x<4\\ x+3, & 4\le x\le 8\end{cases}$

Answer: $k=\dfrac{\ln 7}{4}$

149. $f(x)=\begin{cases}\sqrt{kx}, & 0\le x\le 3\\ x+1, & 3<x\le 10\end{cases}$

Answer: $k=\dfrac{16}{3}$

In the following exercises, use the Intermediate Value Theorem (IVT).

150. Let $h(x)=\begin{cases}3x^2-4, & x\le 2\\ 5+4x, & x>2\end{cases}$ Over the interval $[0,4]$, there is no value of x such that $h(x)=10$, although $h(0)<10$ and $h(4)>10$. Explain why this does not contradict the IVT.

Answer: h is not continuous at $x=2$ since $\lim\limits_{x\to 2^-}h(x)=8\neq\lim\limits_{x\to 2^+}h(x)=13$; the discontinuity is jump thereby "omitting" the values $8\le y<13$.

151. A particle moving along a line has at each time t a position function $s(t)$, which is continuous. Assume $s(2)=5$ and $s(5)=2$. Another particle moves such that its position is given by $h(t)=s(t)-t$. Explain why there must be a value c for $2<c<5$ such that $h(c)=0$.

Answer: Since both s and $y=t$ are continuous everywhere, then $h(t)=s(t)-t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2,5]$. Also, $h(2)=3>0$ and $h(5)=-3<0$. Therefore, by the IVT, there is a value $x=c$ such that $h(c)=0$.

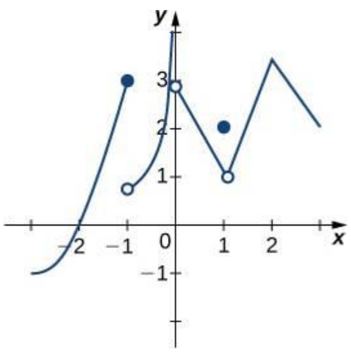
152. **[T]** Use the statement "The cosine of *t* is equal to *t* cubed."
   a. Write a mathematical equation of the statement.
   b. Prove that the equation in part a. has at least one real solution.
   c. Use a calculator to find an interval of length 0.01 that contains a solution.

Answer: a. $\cos t = t^3$; b. Let $f(t) = \cos t - t^3$ and $I = \left[0, \frac{\pi}{2}\right]$. Then, *f* is continuous on *I* and $f(0) = 1 > 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} < 0$, so the IVT applies. That is, there is a value $x = c \in I$ such that $f(c) = 0$. In other words, $\cos c - c^3 = 0 \to \cos c = c^3$. c. $[0.86, 0.87]$

153. Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Answer: The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

154. Consider the graph of the function $y = f(x)$ shown in the following graph.

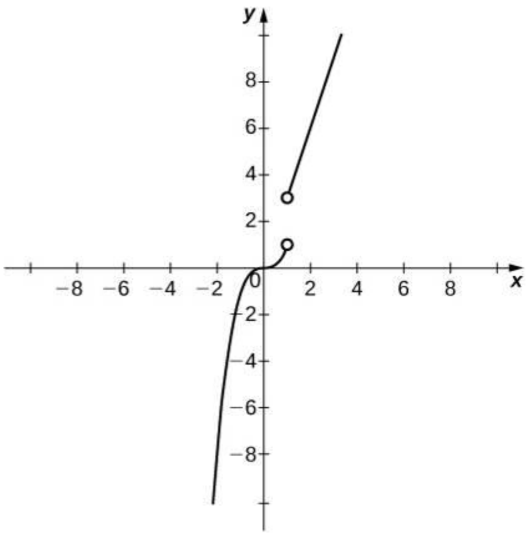


   a. Find all values for which the function is discontinuous.
   b. For each value in part a., state why the formal definition of continuity does not apply.
   c. Classify each discontinuity as either jump, removable, or infinite.

Answer: a. $x = -1, 0, 1$. b. For $x = -1$, $\lim_{x \to -1} f(x)$ does not exist; for $x = 0$, $\lim_{x \to 0} f(x)$ does not exist and $f(0)$ is undefined; for $x = 1$, $\lim_{x \to 1} f(x) \neq f(1)$. c. $x = -1$, jump; $x = 0$, infinite; $x = 1$, removable.

155. Let $f(x) = \begin{cases} 3x, x > 1 \\ x^3, x < 1 \end{cases}$.
   a. Sketch the graph of *f*.
   b. Is it possible to find a value *k* such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer: a.



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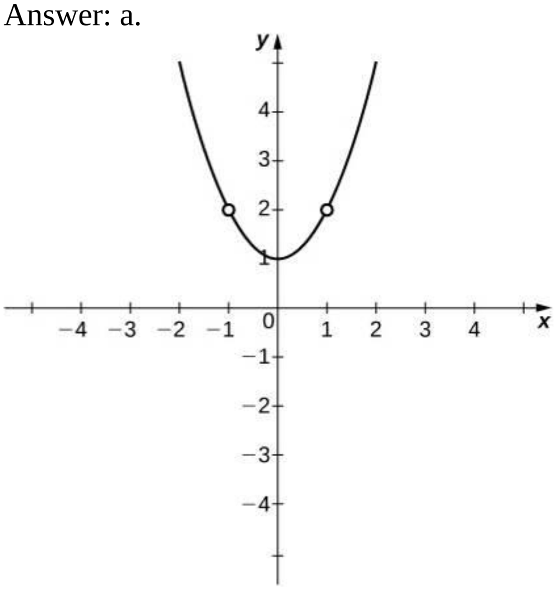
b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

156. Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of *f*.

b. Is it possible to find values k_1 and k_2 such that $f(-1) = k$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer: a.



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b. It is possible to redefine both as $f(-1) = f(1) = 2$ since both discontinuities are removable.

157. Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of *f* is $(-\infty, +\infty)$.

ii. *f* has an infinite discontinuity at $x = -6$.

iii. $f(-6) = 3$

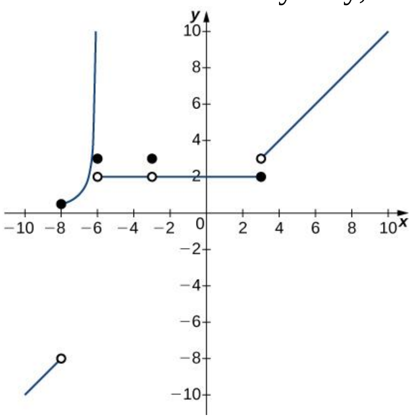
iv. $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$

v. $f(-3) = 3$

vi. *f* is left continuous but not right continuous at $x = 3$.

vii. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

Answer: Answers may vary; see the following example:



158. Sketch the graph of the function $y = f(x)$ with properties i. through iv.

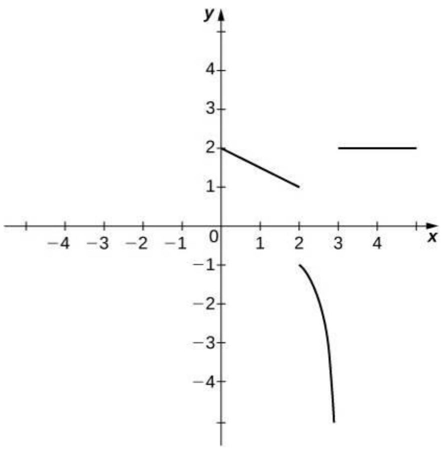
i. The domain of *f* is $[0, 5]$.

ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.

iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

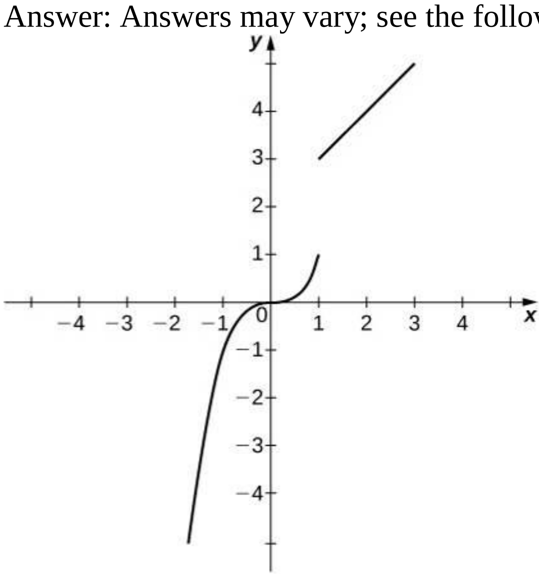
Answer: Answers may vary; see the following example:



In the following exercises, suppose $y = f(x)$ is defined for all *x*. For each description, sketch a graph with the indicated property.

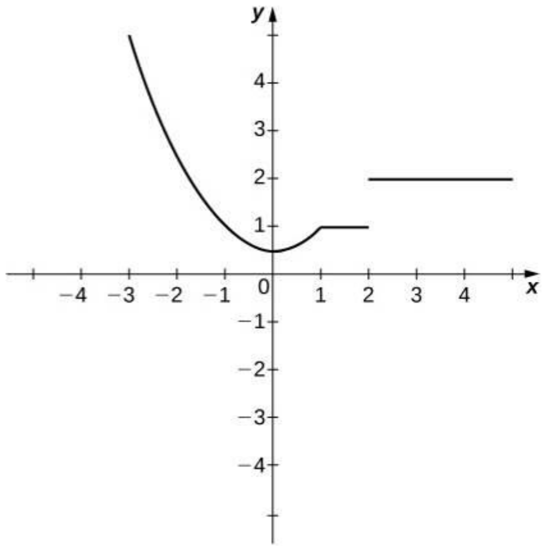
159. Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

Answer: Answers may vary; see the following example:



160. Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Answer: Answers may vary; see the following example:



y
x
4
3
2
1
0
−1
−2
−3
−4
1
2
3
4

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

161. $f(t)=\dfrac{2}{e^t-e^{-t}}$ is continuous everywhere.

Answer: False. It is continuous over $(-\infty,0)\cup(0,\infty)$.

162. If the left- and right-hand limits of $f(x)$ as $x\to a$ exist and are equal, then *f* cannot be discontinuous at $x=a$.

Answer: False. Consider $f(x)=\begin{cases} x \text{ if } x\neq 0 \\ 4 \text{ if } x=0 \end{cases}$.

163. If a function is not continuous at a point, then it is not defined at that point.

Answer: False. Consider $f(x)=\begin{cases} x \text{ if } x\neq 0 \\ 4 \text{ if } x=0 \end{cases}$.

164. According to the IVT, $\cos x-\sin x-x=2$ has a solution over the interval $[-1,1]$.

Answer: True. The function $f(x)=\cos x-\sin x-x-2$ is continuous, and $f(-1)>0$ and $f(1)<0$.

165. If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x)=0$ has exactly one solution in $[a,b]$.

Answer: False. IVT only says that there is *at least* one solution; it does not guarantee that there is exactly one. Consider $f(x)=\cos(x)$ on $[-\pi,2\pi]$.

166. The function $f(x)=\dfrac{x^2-4x+3}{x^2-1}$ is continuous over the interval $[0,3]$.

Answer: False. It has a removable discontinuity at $x=1$.

167. If $f(x)$ is continuous everywhere and $f(a),f(b)>0$, then there is no root of $f(x)$ in the interval $[a,b]$.

Answer: False. The IVT does *not* work in reverse! Consider $(x-1)^2$ over the interval $[-2,2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and *r* is the distance between the two particles.

168. To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.
   a. Explain the physical reasoning behind this assumption.
   b. What is the force equation?
   c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9 \ \text{Nm}^2/\text{C}^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?
   d. Is there any finite value of R for which this system remains continuous at R?

Answer: a. The force between two charged particles at a large distance apart, such as one in the United States and the other in Australia, have a *very* negligible effect on each other. b. $F(r) = \begin{cases} k_e \frac{|q_1 q_2|}{r^2} & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases}$. c. $F(1) = 2.307 \times 10^{-28}$ versus 0. They are very close. d. No.

169. Instead of making the force 0 at R, instead we let the force be 10^{-20} for $r \geq R$. Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9 \ \text{Nm}^2/\text{C}^2$. Is there a value R that can make this system continuous? If so, find it.

Answer: $R = 0.0001519$ m

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where *m* is the mass of the rocket, *d* is the distance of the rocket from the center of Earth, and *k* is a constant.

170. **[T]** Determine the value and units of k given that the mass of the rocket is 3 million kg. (*Hint*: The distance from the center of Earth to its surface is 6378 km.)

Answer: $k = 398,653 \ \text{km}^2/\text{kg}$

171. **[T]** After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F(d) = \begin{cases} -\frac{mk}{d^2} & \text{if } d < D \\ 10,000 & \text{if } d \geq D \end{cases}$. Using the value of k found in the previous exercise, find the necessary condition D such that the force function remains continuous.

Answer: $D = 345{,}826\text{ km}$

172. As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as $F(d) = \begin{cases} -\dfrac{m_1 k}{d^2} & \text{if } d < D \\ -\dfrac{m_2 k}{d^2} & \text{if } d \geq D \end{cases}$. Is there a D value such that this function is continuous, assuming $m_1 \neq m_2$?

Answer: No. It is a jump discontinuity.

Prove the following functions are continuous everywhere

173. $f(\theta) = \sin\theta$

Answer: For all values of a, $f(a)$ is defined, $\lim\limits_{\theta \to a} f(\theta)$ exists, and $\lim\limits_{\theta \to a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

174. $g(x) = |x|$

Answer: $g(x)$ can be rewritten as $g(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$. Since $g(x)$ is continuous at $x = 0$, and the functions $-x$ and x are continuous for all values of x, $g(x)$ is continuous everywhere.

175. Where is $f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$ continuous?

Answer: Nowhere