Chapter 2
Limits
2.5 The Precise Definition of a Limit

Section Exercises

In the following exercises, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

176. $\lim_{x \to a} f(x) = N$

Answer: For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|f(x) - N| < \varepsilon$]

177. $\lim_{t \to b} g(t) = M$

Answer: For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$]

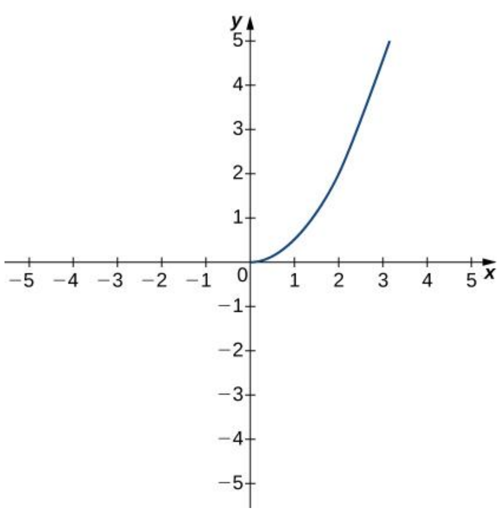
178. $\lim_{x \to c} h(x) = L$

Answer: For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - c| < \delta$, then $|h(x) - L| < \varepsilon$]

179. $\lim_{x \to a} \varphi(x) = A$

Answer: For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$]

The following graph of the function *f* satisfies $\lim_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.

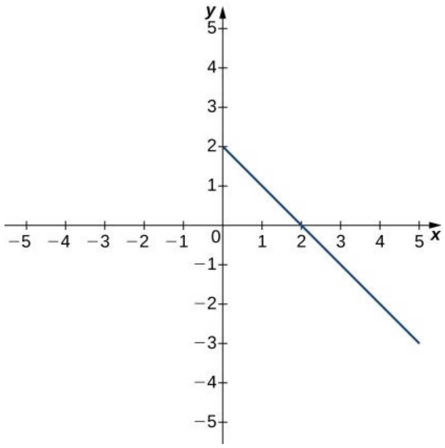


180. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

Answer: $\delta \le 0.5$

181. If $0 < |x-2| < \delta$, then $|f(x)-2| < 0.5$.
Answer: $\delta \le 0.25$

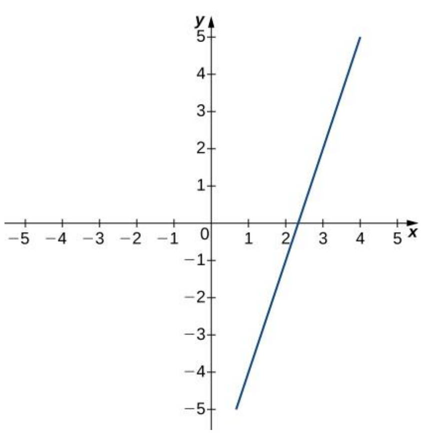
The following graph of the function *f* satisfies $\lim_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



182. If $0 < |x-3| < \delta$, then $|f(x)+1| < 1$.
Answer: $\delta \le 1$

183. If $0 < |x-3| < \delta$, then $|f(x)+1| < 2$.
Answer: $\delta \le 2$

The following graph of the function *f* satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of *ε*, find a value of $\delta > 0$ such that the precise definition of limit holds true.



184. $\varepsilon = 1.5$
Answer: $\delta \le 0.5$

185. $\varepsilon = 3$
Answer: $\delta \le 1$

[T] In the following exercises, use a graphing calculator to find a number δ such that the statements hold true.

186. $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$
Answer: $\delta < 0.2019$

187. $\left|\sqrt{x-4} - 2\right| < 0.1$,whenever $|x - 8| < \delta$
Answer: $\delta < 0.3900$

In the following exercises, use the precise definition of limit to prove the given limits.

188. $\lim_{x \to 2} (5x + 8) = 18$
Answer: Let $\delta = \varepsilon/5$. If $0 < |x - 2| < \varepsilon/5$, then $|5x + 8 - 18| = 5|x - 2| < \varepsilon$.

189. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$
Answer: Let $\delta = \varepsilon$. If $0 < |x - 3| < \varepsilon$, then $|x + 3 - 6| = |x - 3| < \varepsilon$.

190. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2} = 5$
Answer: Let $\delta = \varepsilon/2$.. If $0 < |x - 2| < \varepsilon/2$, then $|2x + 1 - 5| = 2|x - 2| < \varepsilon$.

191. $\lim_{x \to 0} x^4 = 0$
Answer: Let $\delta = \sqrt[4]{\varepsilon}$. If $0 < |x| < \sqrt[4]{\varepsilon}$, then $\left|x^4\right| = x^4 < \varepsilon$.

192. $\lim_{x \to 2} (x^2 + 2x) = 8$
Answer: Let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x - 2| < \delta$ then $\left|x^2 + 2x - 8\right| = |x + 4||x - 2| < 7\delta < \varepsilon$.

In the following exercises, use the precise definition of limit to prove the given one-sided limits.

193. $\lim_{x \to 5^-} \sqrt{5 - x} = 0$
Answer: Let $\delta = \varepsilon^2$. If $5 - \varepsilon^2 < x < 5$, then $\left|\sqrt{5 - x}\right| = \sqrt{5 - x} < \varepsilon$.

194. $\lim_{x \to 0^+} f(x) = -2$, where $f(x) = \begin{cases} 8x-3, & \text{if } x < 0 \\ 4x-2, & \text{if } x \geq 0 \end{cases}$.

Answer: Let $\delta = \varepsilon/4$. If $0 < x < \varepsilon/4$, then $\left|f(x)+2\right| = 4x < \varepsilon$.

195. $\lim_{x \to 1^-} f(x) = 3$, where $f(x) = \begin{cases} 5x-2, & \text{if } x < 1 \\ 7x-1, & \text{if } x \geq 1 \end{cases}$.

Answer: Let $\delta = \varepsilon/5$. If $1-\varepsilon/5 < x < 1$, then $\left|f(x)-3\right| = 5x-5 < \varepsilon$.

In the following exercises, use the precise definition of limit to prove the given infinite limits.

196. $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Answer: Let $\delta = 1/\sqrt{N}$. If $0 < |x| < 1/\sqrt{N}$, then $f(x) = 1/x^2 > N$.

197. $\lim_{x \to -1} \frac{3}{(x+1)^2} = \infty$

Answer: Let $\delta = \sqrt{\frac{3}{M}}$. If $0 < |x+1| < \sqrt{\frac{3}{M}}$, then $f(x) = \frac{3}{(x+1)^2} > M$.

198. $\lim_{x \to 2} -\frac{1}{(x-2)^2} = -\infty$

Answer: Let $\delta = 1/\sqrt{N}$. If $0 < |x-2| < 1/\delta = 1/\sqrt{N}$, then $f(x) = -1/(x-2)^2 > N$.

199. An engineer is using a machine to cut a flat square of Aerogel of area 144 cm^2. If there is a maximum error tolerance in the area of 8 cm^2, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to *δ*, *ε*, *a*, and *L*?

Answer: 0.328 cm, $\varepsilon = 8$, $\delta = 0.33$, $a = 12$, $L = 144$

200. Use the precise definition of limit to prove that the following limit does not exist: $\lim_{x \to 1} \frac{|x-1|}{x-1}$.

Answer: $\varepsilon = 1/2$, $x = 2 \pm \delta/2$

201. Using precise definitions of limits, prove that $\lim_{x \to 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (*Hint*: Try any $\delta < 1$.)

Answer: Answers may vary.

202. Using precise definitions of limits, prove that $\lim_{x\to 0} f(x)$ does not exist:

$f(x)=\begin{cases}1 \text{ if } x \text{ is rational}\\ 0 \text{ if } x \text{ is irrational}\end{cases}$. (*Hint*: Think about how you can always choose a rational number $0<r<d$, but $|f(r)-0|=1$.)

Answer: Answers may vary.

203. Using precise definitions of limits, determine $\lim_{x\to 0} f(x)$ for $f(x)=\begin{cases}x \text{ if } x \text{ is rational}\\ 0 \text{ if } x \text{ is irrational}\end{cases}$.

(*Hint*: Break into two cases, *x* rational and *x* irrational.)

Answer: 0

204. Using the function from the previous exercise, use the precise definition of limits to show that $\lim_{x\to a} f(x)$ does not exist for $a\neq 0$.

Answer: Answers may vary.

For the following exercises, suppose that $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$ both exist. Use the precise definition of limits to prove the following limit laws:

205. $\lim_{x\to a}\left(f(x)+g(x)\right)=L+M$

$$\begin{aligned}&\lim_{x\to a}\left(f(x)+g(x)\right)\\ \text{Answer: } &=\lim_{x\to a}\left(f(x)\right)+\lim_{x\to a}\left(g(x)\right)\\ &=L+M\end{aligned}$$

206. $\lim_{x\to a}\left[cf(x)\right]=cL$ for any real constant *c* (*Hint*: Consider two cases: $c=0$ and $c\neq 0$.)

Answer: Answers may vary.

207. $\lim_{x\to a}\left[f(x)g(x)\right]=LM$. (*Hint*:

$|f(x)g(x)-LM|=|f(x)g(x)-f(x)M+f(x)M-LM|\leq|f(x)||g(x)-M|+|M||f(x)-L|$

.)

Answer: Answers may vary.

Chapter Review Exercises

***True or False*. In the following exercises, justify your answer with a proof or a counterexample.**

208. A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

Answer: False. We also need $\lim_{x \to a} f(x) = f(a)$.

209. You can use the quotient rule to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

Answer: False

210. If there is a vertical asymptote at $x = a$ for the function $f(x)$, then *f* is undefined at the point $x = a$.

Answer: False

211. If $\lim_{x \to a} f(x)$ does not exist, then *f* is undefined at the point $x = a$.

Answer: False. A removable discontinuity is possible.

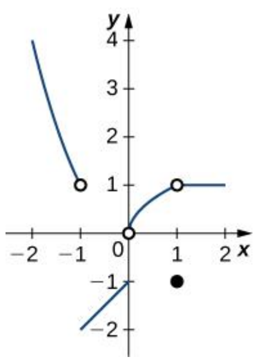
212. Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



Answer: a. DNE; b. 1; c. 0; d. DNE

In the following exercises, evaluate the limit algebraically or explain why the limit does not exist.

213. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$

Answer: 5

214. $\lim_{x\to 0} 3x^2 - 2x + 4$

Answer: 4

215. $\lim_{x\to 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

Answer: $8/7$

216. $\lim_{x\to \pi/2} \frac{\cot x}{\cos x}$

Answer: 1

217. $\lim_{x\to -5} \frac{x^2 + 25}{x + 5}$

Answer: DNE

218. $\lim_{x\to 2} \frac{3x^2 - 2x - 8}{x^2 - 4}$

Answer: $5/2$

219. $\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1}$

Answer: $2/3$

220. $\lim_{x\to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$

Answer: 4

221. $\lim_{x\to 4} \frac{4 - x}{\sqrt{x} - 2}$

Answer: –4

222. $\lim_{x\to 4} \frac{1}{\sqrt{x} - 2}$

Answer: DNE

In the following exercises, use the squeeze theorem to prove the limit.

223. $\lim_{x\to 0} x^2 \cos\left(2\pi x\right) = 0$

Answer: Since $-1 \le \cos(2\pi x) \le 1$, then $-x^2 \le x^2 \cos(2\pi x) \le x^2$. Since $\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} -x^2$, it follows that $\lim_{x\to 0} x^2 \cos\left(2\pi x\right) = 0$.

224. $\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$

Answer: Since $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$, then $-x^3 \le x^3 \sin\left(\frac{\pi}{x}\right) \le x^3$ for $x \ge 0$ and $x^3 \le x^3 \sin\left(\frac{\pi}{x}\right) \le -x^3$ for $x \le 0$. Since $\lim_{x\to 0} x^3 = 0 = \lim_{x\to 0} -x^3$, it follows that $\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$.

225. Determine the domain such that the function $f(x) = \sqrt{x-2} + xe^x$ is continuous over its domain.
Answer: $[2,\infty]$

In the following exercises, determine the value of *c* such that the function remains continuous. Draw your resulting function to ensure it is continuous.

226. $f(x) = \begin{cases} x^2+1, & x > c \\ 2x, & x \le c \end{cases}$

Answer: $c = 1$

227. $f(x) = \begin{cases} \sqrt{x+1}, & x > -1 \\ x^2 + c, & x \le -1 \end{cases}$

Answer: $c = -1$

In the following exercises, use the precise definition of limit to prove the limit.

228. $\lim_{x\to 1}(8x+16) = 24$

Answer: $\delta = \varepsilon/8$

229. $\lim_{x\to 0} x^3 = 0$

Answer: $\delta = \sqrt[3]{\varepsilon}$

230. A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

Answer: Since $x(5) > 0$, $x(6) < 0$, and $x(t)$ for all values of t, the Intermediate Value Theorem states that there is some value of t in $(5,6)$ where $x(t) = 0$.

231. A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where *x* is measured in meters and *t* is measured in seconds. Find the average velocity over the time period $t = [0,2]$.

Answer: 0 m/sec

232. From the previous exercises, estimate the instantaneous velocity at $t=2$ by checking the average velocity within $t=0.01$ sec.

Answer: Between 1.99 m/sec and 2.01 m/sec.