Chapter 2
Limits
2.3 The Limit Laws

Section Exercises

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

83. $\lim_{x\to 0}\left(4x^2-2x+3\right)$

Answer: Use constant multiple law and difference law:

$\lim_{x\to 0}\left(4x^2-2x+3\right)=4\lim_{x\to 0}x^2-2\lim_{x\to 0}x+\lim_{x\to 0}3=3$

84. $\lim_{x\to 1}\frac{x^3+3x^2+5}{4-7x}$

Answer: Use quotient law: $\lim_{x\to 1}\frac{x^3+3x^2+5}{4-7x}=\frac{\lim_{x\to 1}\left(x^3+3x^2+5\right)}{\lim_{x\to 1}\left(4-7x\right)}=-3$

85. $\lim_{x\to -2}\sqrt{x^2-6x+3}$

Answer: Use root law: $\lim_{x\to -2}\sqrt{x^2-6x+3}=\sqrt{\lim_{x\to -2}\left(x^2-6x+3\right)}=\sqrt{19}$

86. $\lim_{x\to -1}\left(9x+1\right)^2$

Answer: Use power law: $\lim_{x\to -1}\left(9x+1\right)^2=\left(\lim_{x\to -1}\left(9x+1\right)\right)^2=64$

In the following exercises, use direct substitution to evaluate each limit.

87. $\lim_{x\to 7}x^2$

Answer: 49

88. $\lim_{x\to -2}\left(4x^2-1\right)$

Answer: 15

89. $\lim_{x\to 0}\frac{1}{1+\sin x}$

Answer: 1

90. $\lim\limits_{x\to 2} e^{2x-x^2}$

Answer: 1

91. $\lim\limits_{x\to 1} \dfrac{2-7x}{x+6}$

Answer: $-\dfrac{5}{7}$

92. $\lim\limits_{x\to 3} \ln e^{3x}$

Answer: 9

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

93. $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4}$

Answer: $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} = \dfrac{16-16}{4-4} = \dfrac{0}{0}$; then, $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} = \lim\limits_{x\to 4} \dfrac{(x+4)(x-4)}{x-4} = 8$

94. $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-2x}$

Answer: $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-2x} = \dfrac{2-2}{4-4} = \dfrac{0}{0}$; then, $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-2x} = \lim\limits_{x\to 2} \dfrac{x-2}{x(x-2)} = \dfrac{1}{2}$

95. $\lim\limits_{x\to 6} \dfrac{3x-18}{2x-12}$

Answer: $\lim\limits_{x\to 6} \dfrac{3x-18}{2x-12} = \dfrac{18-18}{12-12} = \dfrac{0}{0}$; then, $\lim\limits_{x\to 6} \dfrac{3x-18}{2x-12} = \lim\limits_{x\to 6} \dfrac{3(x-6)}{2(x-6)} = \dfrac{3}{2}$

96. $\lim\limits_{x\to 0} \dfrac{(1+h)^2-1}{h}$

Answer: $\lim\limits_{x\to 0} \dfrac{(1+h)^2-1}{h} = \dfrac{1-1}{0} = \dfrac{0}{0}$; then, $\lim\limits_{x\to 0} \dfrac{(1+h)^2-1}{h} = \lim\limits_{x\to 0} \dfrac{1+2h+h^2-1}{h} = \lim\limits_{x\to 0} \dfrac{h(h+2)}{h} = 2$

97. $\lim\limits_{x\to 9} \dfrac{t-9}{\sqrt{t}-3}$

Answer: $\lim\limits_{x\to 9} \dfrac{t-9}{\sqrt{t}-3} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$; then, $\lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3} = \lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3} \dfrac{\sqrt{t}+3}{\sqrt{t}+3} = \lim\limits_{t\to 9} \left(\sqrt{t}+3\right) = 6$

98. $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$, where a is a non-zero real-valued constant

Answer: $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \dfrac{\dfrac{1}{a} - \dfrac{1}{a}}{0} = \dfrac{0}{0}$; then, $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{a-(a+h)}{a(a+h)}}{h} = \lim\limits_{h \to 0} \dfrac{-h}{ha(a+h)} = -\dfrac{1}{a^2}$

99. $\lim\limits_{\theta \to \pi} \dfrac{\sin\theta}{\tan\theta}$

Answer: $\lim\limits_{\theta \to \pi} \dfrac{\sin\theta}{\tan\theta} = \dfrac{\sin\pi}{\tan\pi} = \dfrac{0}{0}$; then, $\lim\limits_{\theta \to \pi} \dfrac{\sin\theta}{\tan\theta} = \lim\limits_{\theta \to \pi} \dfrac{\sin\theta}{\dfrac{\sin\theta}{\cos\theta}} = \lim\limits_{\theta \to \pi} \cos\theta = -1$

100. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

Answer: $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$; then, $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x^2+x+1)}{(x+1)(x-1)} = \dfrac{3}{2}$

101. $\lim\limits_{x \to 1/2} \dfrac{2x^2 + 3x - 2}{2x - 1}$

Answer: $\lim\limits_{x \to 1/2} \dfrac{2x^2 + 3x - 2}{2x - 1} = \dfrac{\dfrac{1}{2} + \dfrac{3}{2} - 2}{1-1} = \dfrac{0}{0}$; then, $\lim\limits_{x \to 1/2} \dfrac{2x^2 + 3x - 2}{2x - 1} = \lim\limits_{x \to 1/2} \dfrac{(2x-1)(x+2)}{2x-1} = \dfrac{5}{2}$

102. $\lim\limits_{x \to -3} \dfrac{\sqrt{x+4} - 1}{x+3}$

Answer: $\lim\limits_{x \to -3} \dfrac{\sqrt{x+4} - 1}{x+3} = \dfrac{1-1}{-3+3} = \dfrac{0}{0}$; then,

$$\lim_{x \to -3} \frac{\sqrt{x+4} - 1}{x+3} = \lim_{x \to -3} \frac{\sqrt{x+4} - 1}{x+3} \frac{\sqrt{x+4} + 1}{\sqrt{x+4} + 1} = \lim_{x \to -3} \frac{1}{\sqrt{x+4} + 1} = \frac{1}{2}$$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of Example_02_03_11 to simplify the function to help determine the limit.

103. $\lim\limits_{x \to -2^-} \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$

Answer: $-\infty$

104. $\lim\limits_{x \to -2^+} \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$

Answer: $+\infty$

105. $\lim_{x\to 1^-} \frac{2x^2+7x-4}{x^2+x-2}$

Answer: $-\infty$

106. $\lim_{x\to 1^+} \frac{2x^2+7x-4}{x^2+x-2}$

Answer: $+\infty$

In the following exercises, assume that $\lim_{x\to 6} f(x)=4$, $\lim_{x\to 6} g(x)=9$, and $\lim_{x\to 6} h(x)=6$. Use these three facts and the limit laws to evaluate each limit.

107. $\lim_{x\to 6} 2f(x)g(x)$

Answer: $\lim_{x\to 6} 2f(x)g(x) = 2\lim_{x\to 6} f(x) \lim_{x\to 6} g(x) = 72$

108. $\lim_{x\to 6} \frac{g(x)-1}{f(x)}$

Answer: $\lim_{x\to 6} \frac{g(x)-1}{f(x)} = \frac{\lim_{x\to 6}(g(x))-1}{\lim_{x\to 6} f(x)} = 2$

109. $\lim_{x\to 6} \left(f(x)+\frac{1}{3}g(x) \right)$

Answer: $\lim_{x\to 6} \left(f(x)+\frac{1}{3}g(x) \right) = \lim_{x\to 6} f(x) + \frac{1}{3}\lim_{x\to 6} g(x) = 7$

110. $\lim_{x\to 6} \frac{(h(x))^3}{2}$

Answer: $\lim_{x\to 6} \frac{(h(x))^3}{2} = \frac{1}{2}\left(\lim_{x\to 6} h(x)\right)^3 = 108$

111. $\lim_{x\to 6} \sqrt{g(x)-f(x)}$

Answer: $\lim_{x\to 6} \sqrt{g(x)-f(x)} = \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)} = \sqrt{5}$

112. $\lim_{x\to 6} x\cdot h(x)$

Answer: $\lim_{x\to 6} xh(x) = \left(\lim_{x\to 6} x\right)\left(\lim_{x\to 6} h(x)\right) = 36$

113. $\lim_{x\to 6} \left[(x+1)\cdot f(x)\right]$

Answer: $\lim_{x\to 6}\left[(x+1)f(x)\right]=\left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6}f(x)\right)=28$

114. $\lim_{x\to 6}\left(f(x)\cdot g(x)-h(x)\right)$

Answer: $\lim_{x\to 6}\left(f(x)g(x)-h(x)\right)=\left(\lim_{x\to 6}f(x)\right)\left(\lim_{x\to 6}g(x)\right)-\lim_{x\to 6}h(x)=30$

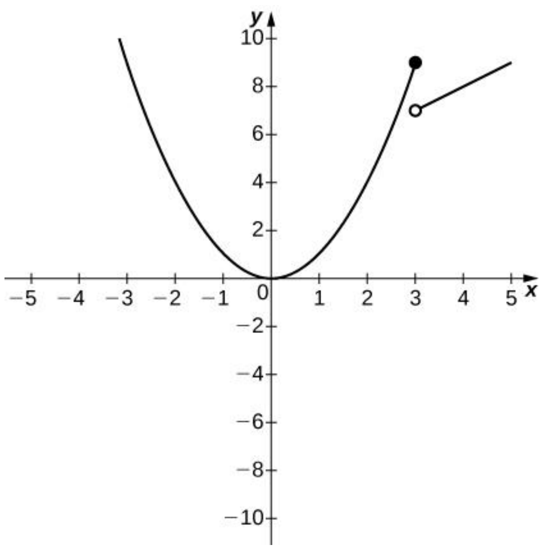
[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

115. $f(x)=\begin{cases}x^2, & x\le 3\\ x+4, & x>3\end{cases}$

a. $\lim_{x\to 3^-}f(x)$

b. $\lim_{x\to 3^+}f(x)$

Answer:



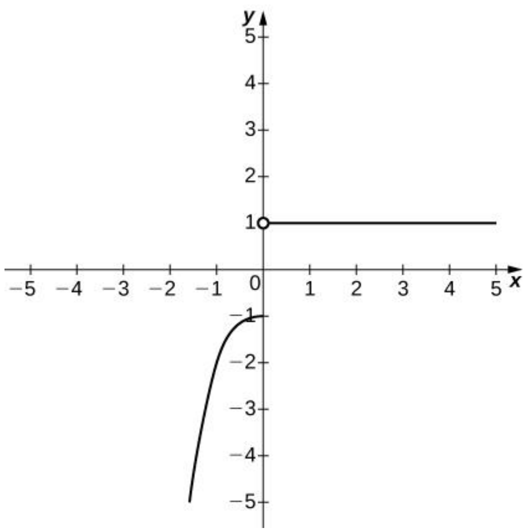
a. 9; b. 7

116. $g(x)=\begin{cases} x^3-1, & x\le 0 \\ 1, & x>0 \end{cases}$

a. $\lim_{x\to 0^-} g(x)$

b. $\lim_{x\to 0^+} g(x)$

Answer:



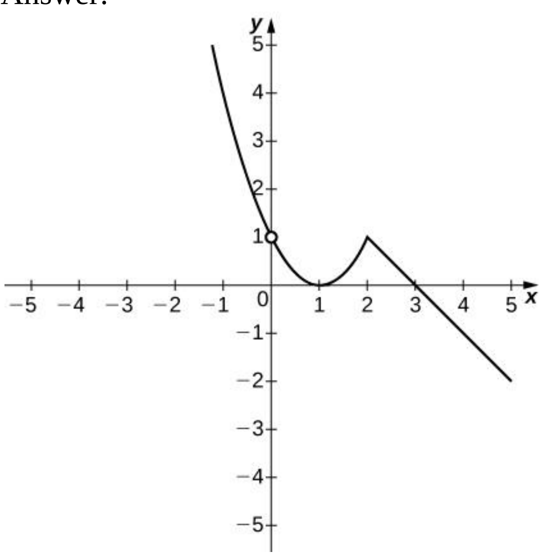
a. –1; b. 1

117. $h(x)=\begin{cases} x^2-2x+1, & x<2 \\ 3-x, & x\ge 2 \end{cases}$

a. $\lim_{x\to 2^-} h(x)$

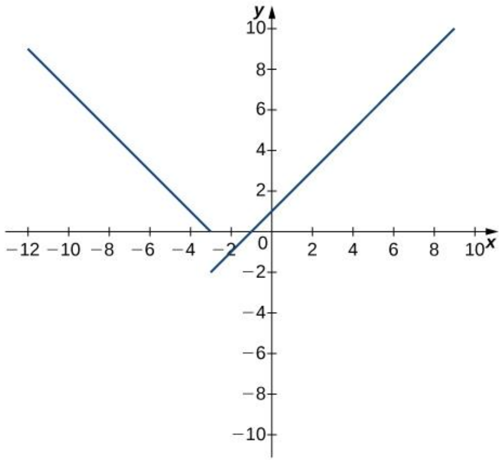
b. $\lim_{x\to 2^+} h(x)$

Answer:

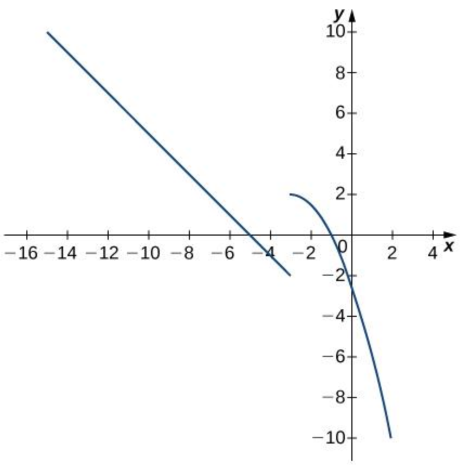


a. 1; b. 1

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

118. $\lim\limits_{x\to-3^+}\left(f(x)+g(x)\right)$

Answer: $\lim\limits_{x\to-3^+}\left(f(x)+g(x)\right)=\lim\limits_{x\to-3^+} f(x)+\lim\limits_{x\to-3^+} g(x)=-2+2=0$

119. $\lim\limits_{x\to-3^-}\left(f(x)-3g(x)\right)$

Answer: $\lim\limits_{x\to-3^-}\left(f(x)-3g(x)\right)=\lim\limits_{x\to-3^-} f(x)-3\lim\limits_{x\to-3^-} g(x)=0+6=6$

120. $\lim\limits_{x\to0}\dfrac{f(x)g(x)}{3}$

Answer: $\lim\limits_{x\to0}\dfrac{f(x)g(x)}{3}=\dfrac{1}{3}\left(\lim\limits_{x\to0} f(x)\right)\left(\lim\limits_{x\to0} g(x)\right)=\dfrac{1}{3}(1)\left(-\dfrac{7}{3}\right)=-\dfrac{7}{9}$

121. $\lim\limits_{x\to-5} \dfrac{2+g(x)}{f(x)}$

Answer: $\lim\limits_{x\to-5} \dfrac{2+g(x)}{f(x)} = \dfrac{2+\left(\lim\limits_{x\to-5} g(x)\right)}{\lim\limits_{x\to-5} f(x)} = \dfrac{2+0}{2} = 1$

122. $\lim\limits_{x\to1} \left(f(x)\right)^2$

Answer: $\lim\limits_{x\to1} \left(f(x)\right)^2 = \left(\lim\limits_{x\to1} f(x)\right)^2 = 2^2 = 4$

123. $\lim\limits_{x\to1} \sqrt{f(x)-g(x)}$

Answer: $\lim\limits_{x\to1} \sqrt[3]{f(x)-g(x)} = \sqrt[3]{\lim\limits_{x\to1} f(x) - \lim\limits_{x\to1} g(x)} = \sqrt[3]{2+5} = \sqrt[3]{7}$

124. $\lim\limits_{x\to-7} \left(x \cdot g(x)\right)$

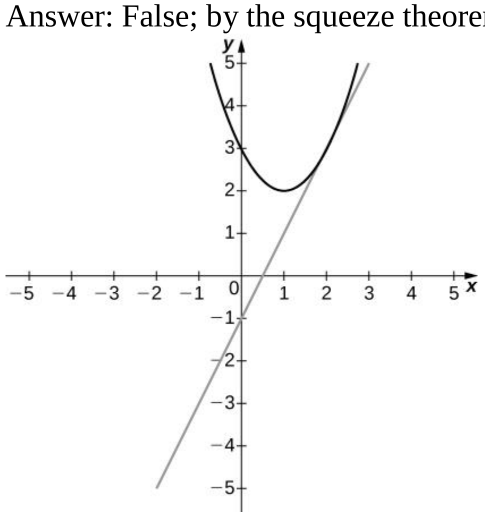
Answer: $\lim\limits_{x\to-7} \left(xg(x)\right) = \left(\lim\limits_{x\to-7}(x)\right)\left(\lim\limits_{x\to-7}\left(g(x)\right)\right) = (-7)(2) = -14$

125. $\lim\limits_{x\to-9} \left[x \cdot f(x) + 2 \cdot g(x)\right]$

Answer: $\lim\limits_{x\to-9} \left(xf(x) + 2g(x)\right) = \left(\lim\limits_{x\to-9} x\right)\left(\lim\limits_{x\to-9} f(x)\right) + 2\lim\limits_{x\to-9}\left(g(x)\right) = (-9)(6) + 2(4) = -46$

126. **[T]** True or False? If $2x-1 \le g(x) \le x^2-2x+3$, then $\lim\limits_{x\to2} g(x) = 0$.

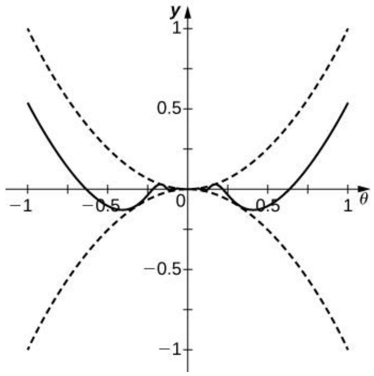
Answer: False; by the squeeze theorem, the limit is 3.



For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

127. **[T]** $\lim\limits_{\theta \to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

Answer: The limit is zero.



128. $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

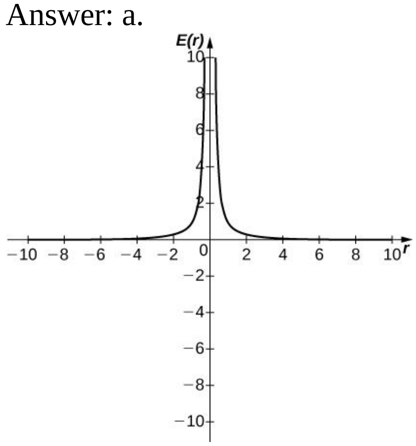
Answer: The limit is zero. (*Hint*: $0 \le f(x) \le x^2$)

129. **[T]** In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 10^9\ \mathrm{N \cdot m^2 / C^2}$.

   a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.

   b. Evaluate $\lim\limits_{r \to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Answer: a.



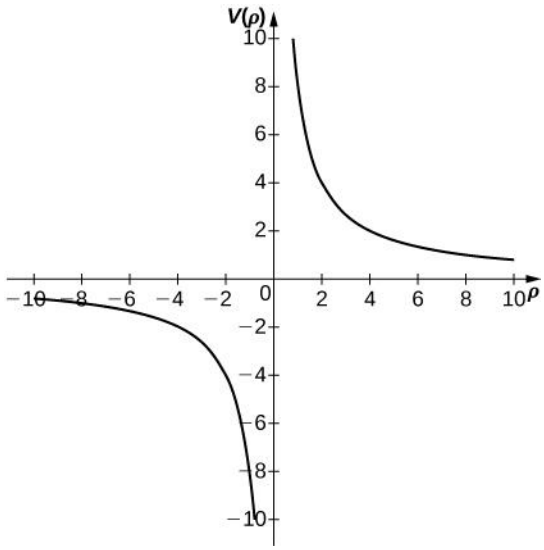
b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

130. **[T]** The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg ($m = 8$).

b. Evaluate $\lim_{x \to 0^+} V(r)$ and explain the physical meaning.

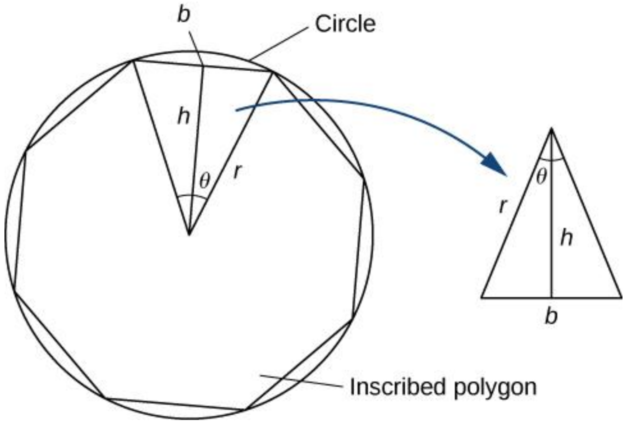
Answer: a.



b. ∞. As the density of the material becomes smaller and smaller, more and more volume is required to reach a mass of 8 kg.

Student Project
Deriving the Formula for the Area of a Circle

1. Express the height h and the base b of the isosceles triangle in the figure below in terms of θ and r.



Answer: 1. $b = 2r\sin\left(\frac{\theta}{2}\right)$ and $h = r\cos\left(\frac{\theta}{2}\right)$

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r. (Substitute $(1/2)\sin q$ for $\sin(q/2)\cos(q/2)$ in your expression.)

Answer: $A = \frac{1}{2}bh = r^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}r^2\sin\theta$

3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)

Answer: $n = \frac{2\pi}{\theta}$

4. Find an expression for the area of the n-sided polygon in terms of r and θ.

Answer: Area of n-sided inscribed polygon $= nA = \frac{\pi r^2\sin\theta}{\theta}$

5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (*Hint:* $\lim\limits_{\theta\to 0}\frac{(\sin\theta)}{\theta} = 1$).

Answer: Area of circle $\lim\limits_{\theta\to 0}\frac{\pi r^2\sin\theta}{\theta} = \pi r^2$