

6 | APPLICATIONS OF INTEGRATION



Figure 6.1 Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia)

Chapter Outline

- 6.1 Areas between Curves
- 6.2 Determining Volumes by Slicing
- 6.3 Volumes of Revolution: Cylindrical Shells
- 6.4 Arc Length of a Curve and Surface Area
- 6.5 Physical Applications
- 6.6 Moments and Centers of Mass
- 6.7 Integrals, Exponential Functions, and Logarithms
- 6.8 Exponential Growth and Decay
- 6.9 Calculus of the Hyperbolic Functions

Introduction

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force (see **Example 6.28**).

Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

6.1 | Areas between Curves

Learning Objectives

- 6.1.1** Determine the area of a region between two curves by integrating with respect to the independent variable.
- 6.1.2** Find the area of a compound region.
- 6.1.3** Determine the area of a region between two curves by integrating with respect to the dependent variable.

In **Introduction to Integration**, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x , beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y .

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

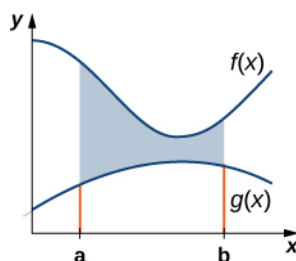


Figure 6.2 The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$.

As we did before, we are going to partition the interval on the x -axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \dots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \dots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. **Figure 6.3(a)** shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. **Figure 6.3(b)** shows a representative rectangle in detail.



Use this **calculator** (http://www.openstax.org//20_CurveCalc) to learn more about the areas between two curves.

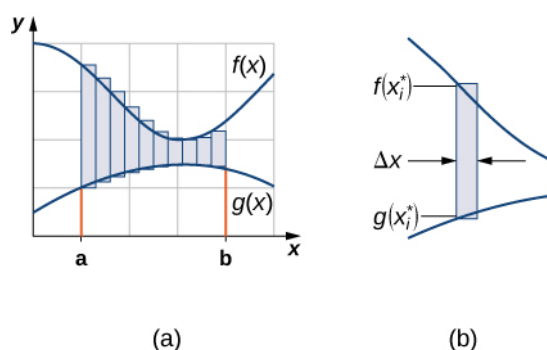


Figure 6.3 (a) We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx . Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \rightarrow \infty$ and we get

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx.$$

These findings are summarized in the following theorem.

Theorem 6.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)] dx. \quad (6.1)$$

We apply this theorem in the following example.

Example 6.1

Finding the Area of a Region between Two Curves 1

If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region R .

Solution

The region is depicted in the following figure.

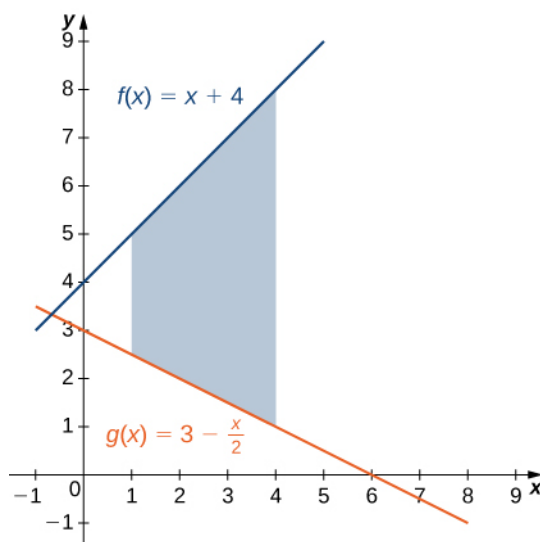


Figure 6.4 A region between two curves is shown where one curve is always greater than the other.

We have

$$\begin{aligned}
 A &= \int_a^b [f(x) - g(x)] dx \\
 &= \int_1^4 \left[(x + 4) - \left(3 - \frac{x}{2} \right) \right] dx = \int_1^4 \left[\frac{3x}{2} + 1 \right] dx \\
 &= \left[\frac{3x^2}{4} + x \right]_1^4 = \left(16 - \frac{7}{4} \right) = \frac{57}{4}.
 \end{aligned}$$

The area of the region is $\frac{57}{4}$ units².



6.1 If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R .

In **Example 6.1**, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 6.2

Finding the Area of a Region between Two Curves 2

If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R .

Solution

The region is depicted in the following figure.

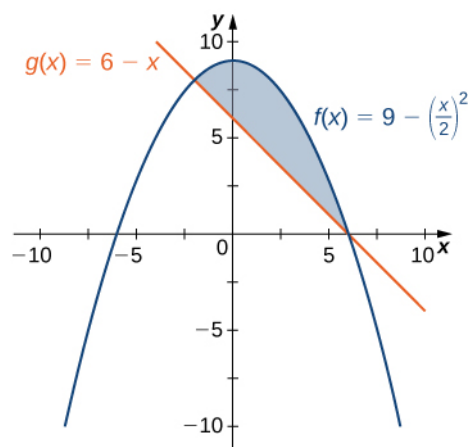


Figure 6.5 This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x - 6)(x + 2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6 . Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^6 \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x) \right] dx = \int_{-2}^6 \left[3 - \frac{x^2}{4} + x \right] dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2} \right]_{-2}^6 = \frac{64}{3}. \end{aligned}$$

The area of the region is $64/3$ units².



6.2 If R is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region R .

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 6.2: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 6.3

Finding the Area of a Region Bounded by Functions That Cross

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R .

Solution

The region is depicted in the following figure.

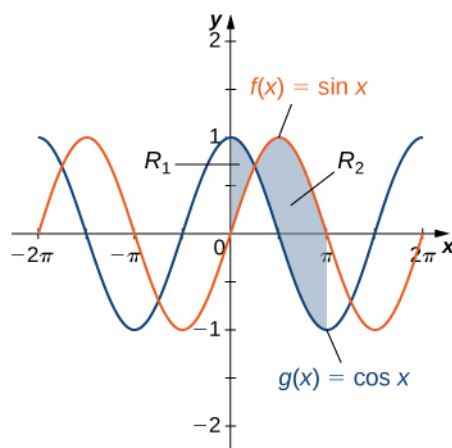


Figure 6.6 The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^{\pi} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ units².



6.3 If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region R .

Example 6.4

Finding the Area of a Complex Region

Consider the region depicted in **Figure 6.7**. Find the area of R .

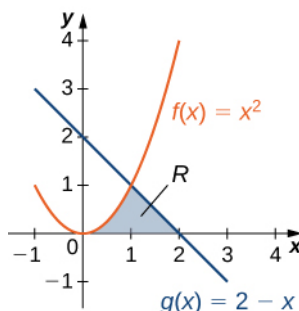


Figure 6.7 Two integrals are required to calculate the area of this region.

Solution

As with **Example 6.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x -axis, so we have

$$A_1 = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x -axis, so we have

$$A_2 = \int_1^2 (2 - x) dx = \left[2x - \frac{x^2}{2} \right]_1^2 = \frac{1}{2}.$$

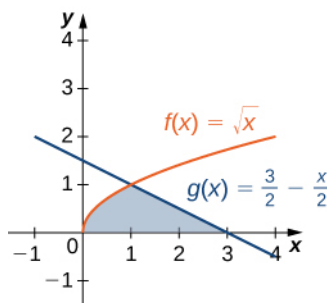
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units².



6.4 Consider the region depicted in the following figure. Find the area of R .



Regions Defined with Respect to y

In **Example 6.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y , instead of as functions of x ?

Review **Figure 6.7**. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y . However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y , we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y , we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

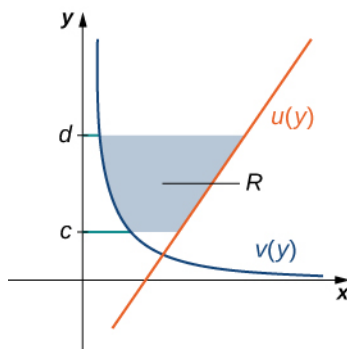


Figure 6.8 We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the y -axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \dots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \dots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. **Figure 6.9(a)** shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$.

Figure 6.9(b) shows a representative rectangle in detail.

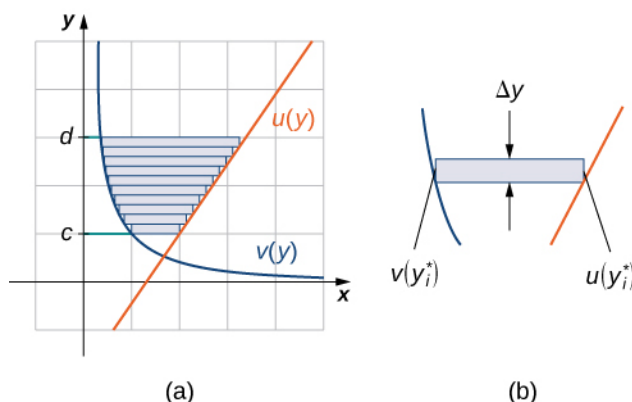


Figure 6.9 (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^n [u(y_i^*) - v(y_i^*)] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \rightarrow \infty$, obtaining

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n [u(y_i^*) - v(y_i^*)] \Delta y = \int_c^d [u(y) - v(y)] dy.$$

These findings are summarized in the following theorem.

Theorem 6.3: Finding the Area between Two Curves, Integrating along the y -axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)] dy. \quad (6.2)$$

Example 6.5

Integrating with Respect to y

Let's revisit **Example 6.4**, only this time let's integrate with respect to y . Let R be the region depicted in **Figure 6.10**. Find the area of R by integrating with respect to y .

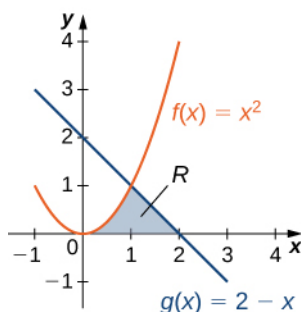


Figure 6.10 The area of region R can be calculated using one integral only when the curves are treated as functions of y .

Solution

We must first express the graphs as functions of y . As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x -axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)] dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}] dy = \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2} \right] \bigg|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units².



6.5 Let's revisit the checkpoint associated with **Example 6.4**, only this time, let's integrate with respect to y . Let R be the region depicted in the following figure. Find the area of R by integrating with respect to y .

