6.9 | Calculus of the Hyperbolic Functions

Learning Objectives

6.9.1 Apply the formulas for derivatives and integrals of the hyperbolic functions.
6.9.2 Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
6.9.3 Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions in **Introduction to Functions and Graphs**, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.

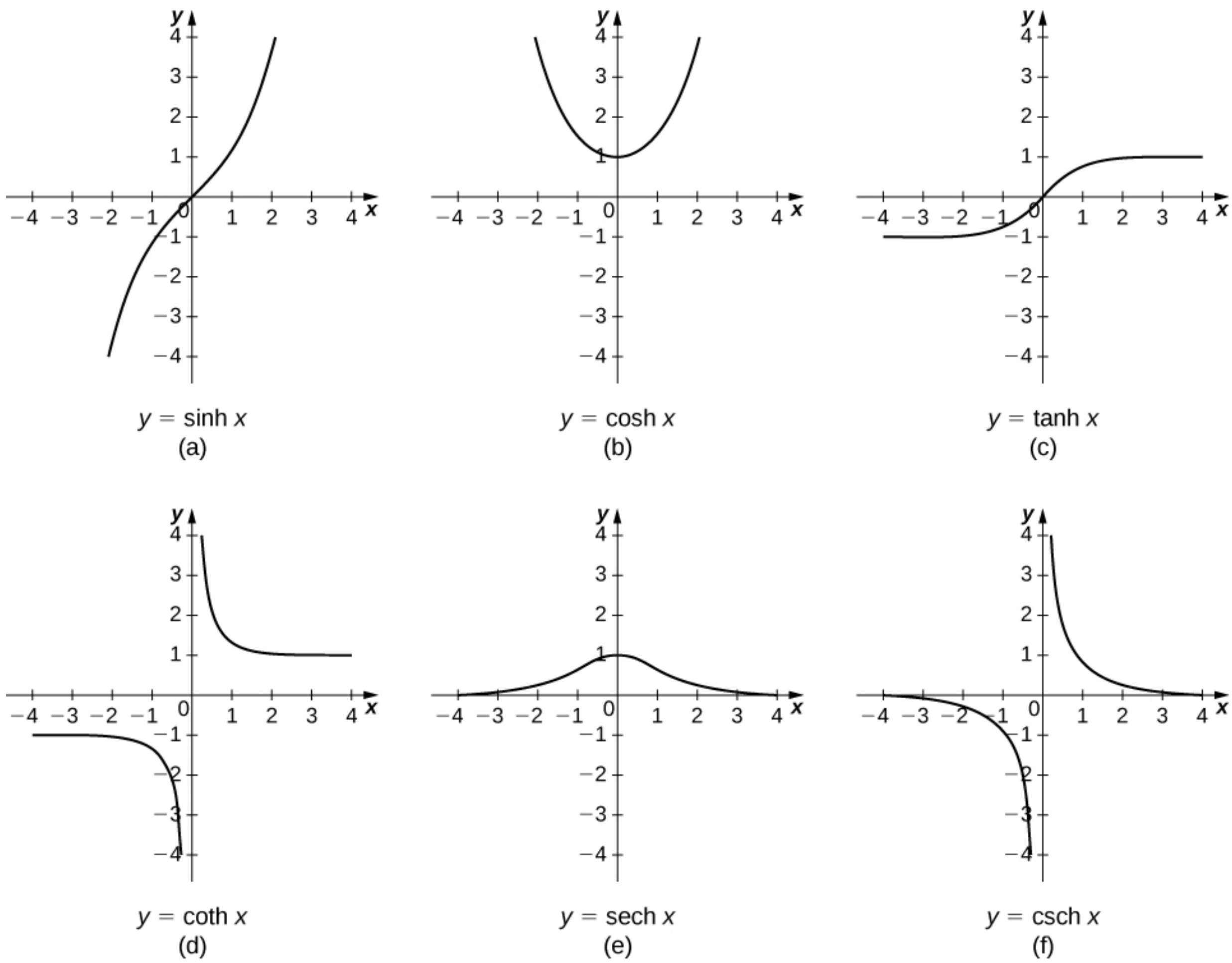

Figure 6.81 Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]\\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x.\end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

Table 6.2 Derivatives of the Hyperbolic Functions

Let’s take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx)\sin x = \cos x$ and $(d/dx)\sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx)\cos x = -\sin x$, but $(d/dx)\cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\begin{aligned}\int \sinh u\,du &= \cosh u + C & \int \operatorname{csch}^2 u\,du &= -\coth u + C\\ \int \cosh u\,du &= \sinh u + C & \int \operatorname{sech} u \tanh u\,du &= -\operatorname{sech} u + C\\ \int \operatorname{sech}^2 u\,du &= \tanh u + C & \int \operatorname{csch} u \coth u\,du &= -\operatorname{csch} u + C\end{aligned}$$

Example 6.47

Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$

b. $\frac{d}{dx}(\cosh x)^2$

Solution

Using the formulas in **Table 6.2** and the chain rule, we get

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right) = \cosh\left(x^2\right) \cdot 2x$

b. $\frac{d}{dx}(\cosh x)^2 = 2\cosh x \sinh x$

6.47 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2 + 3x\right)\right)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Example 6.48

Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x \cosh\left(x^2\right) dx$

b. $\int \tanh x \, dx$

Solution

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\,dx$ and

$$\int x \cosh\left(x^2\right) dx = \int \frac{1}{2}\cosh u \, du = \frac{1}{2}\sinh u + C = \frac{1}{2}\sinh\left(x^2\right) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x \, dx$ and

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain

$$\int \tanh x \, dx = \ln(\cosh x) + C.$$

6.48 Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x \, dx$

b. $\int \operatorname{sech}^2 (3x) dx$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Table 6.3 Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.

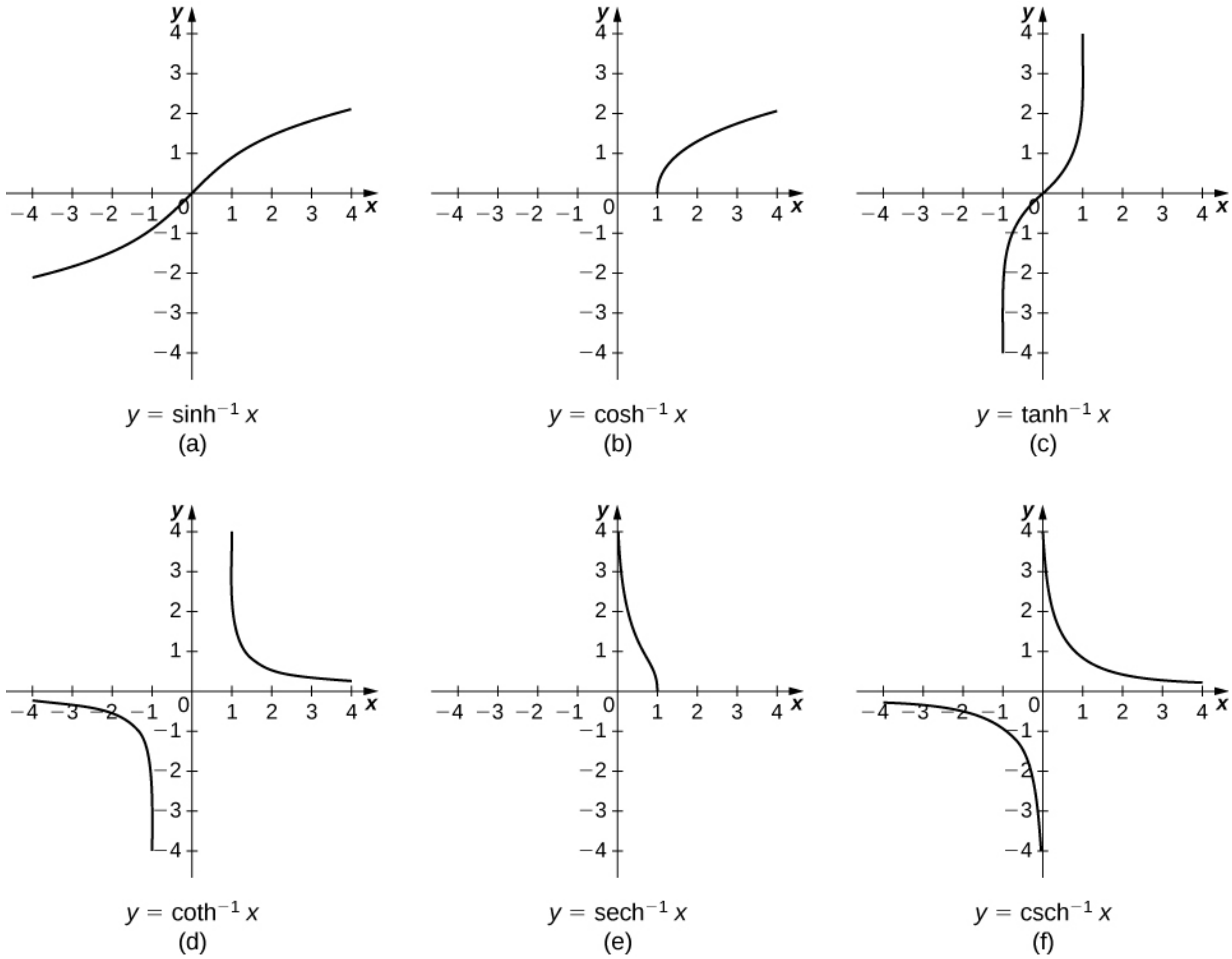

Figure 6.82 Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Table 6.4 Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/\left(1-x^2\right)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\begin{aligned}
\int \frac{1}{\sqrt{1+u^2}}du &= \sinh^{-1} u + C & \int \frac{1}{u\sqrt{1-u^2}}du &= -\operatorname{sech}^{-1} |u| + C \\
\int \frac{1}{\sqrt{u^2-1}}du &= \cosh^{-1} u + C & \int \frac{1}{u\sqrt{1+u^2}}du &= -\operatorname{csch}^{-1} |u| + C \\
\int \frac{1}{1-u^2}du &= \begin{cases} \tanh^{-1} u + C \text{ if } |u| < 1 \\ \coth^{-1} u + C \text{ if } |u| > 1 \end{cases}
\end{aligned}$$

Example 6.49

Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

Solution

Using the formulas in **Table 6.4** and the chain rule, we obtain the following results:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2 = \frac{2\left(\tanh^{-1} x\right)}{1-x^2}$

6.49 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)$

b. $\frac{d}{dx}\left(\coth^{-1} x\right)^3$

Example 6.50

Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution

We can use u-substitution in both cases.

a. Let $u = 2x$. Then, $du = 2dx$ and we have

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du = \frac{1}{2}\cosh^{-1} u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x$. Then, $du = 3dx$ and we obtain

$$\int \frac{1}{2x\sqrt{1-9x^2}}dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du = -\frac{1}{2}\text{sech}^{-1}|u| + C = -\frac{1}{2}\text{sech}^{-1}|3x| + C.$$

6.50 Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, \quad x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 6.83 Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh(x/a)$ are catenaries. **Figure 6.84** shows the graph of $y = 2\cosh(x/2)$.

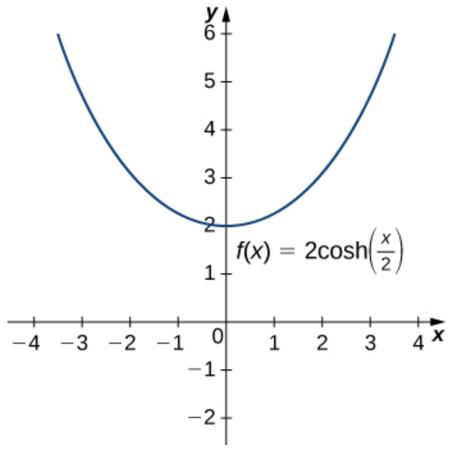

Figure 6.84 A hyperbolic cosine function forms the shape of a catenary.

Example 6.51

Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 2.4 that the formula for arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx.\end{aligned}$$

Now recall that $1 + \sinh^2 x = \cosh^2 x$, so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right) dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] = 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586\text{ ft.}\end{aligned}$$

6.51 Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).