5.2 EXERCISES

In the following exercises, express the limits as integrals.

60. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} (x_i^*) \Delta x$ over $[1, 3]$

61. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(5(x_i^*)^2 - 3(x_i^*)^3\right) \Delta x$ over $[0, 2]$

62. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \sin^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

63. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \cos^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

In the following exercises, given L_n or R_n as indicated, express their limits as $n \to \infty$ as definite integrals, identifying the correct intervals.

64. $L_n = \frac{1}{n} \sum\limits_{i=1}^{n} \frac{i-1}{n}$

65. $R_n = \frac{1}{n} \sum\limits_{i=1}^{n} \frac{i}{n}$

66. $L_n = \frac{2}{n} \sum\limits_{i=1}^{n} \left(1 + 2\frac{i-1}{n}\right)$

67. $R_n = \frac{3}{n} \sum\limits_{i=1}^{n} \left(3 + 3\frac{i}{n}\right)$

68. $L_n = \frac{2\pi}{n} \sum\limits_{i=1}^{n} 2\pi \frac{i-1}{n} \cos\left(2\pi \frac{i-1}{n}\right)$

69. $R_n = \frac{1}{n} \sum\limits_{i=1}^{n} \left(1 + \frac{i}{n}\right) \log\left(\left(1 + \frac{i}{n}\right)^2\right)$

In the following exercises, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the *x*-axis.

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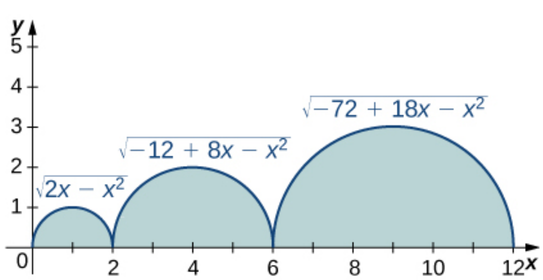

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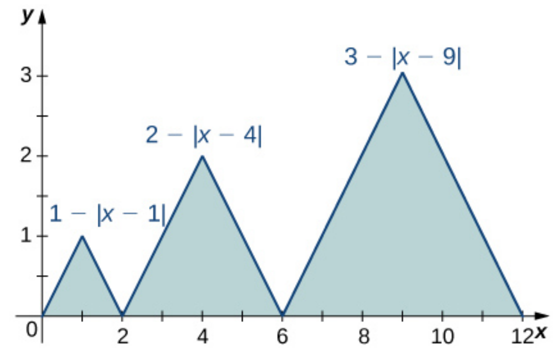

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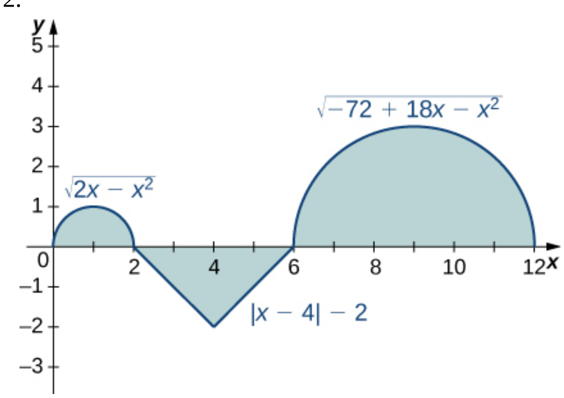

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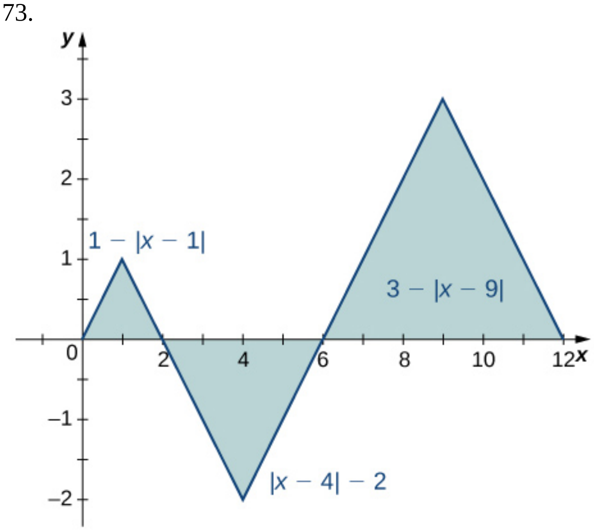

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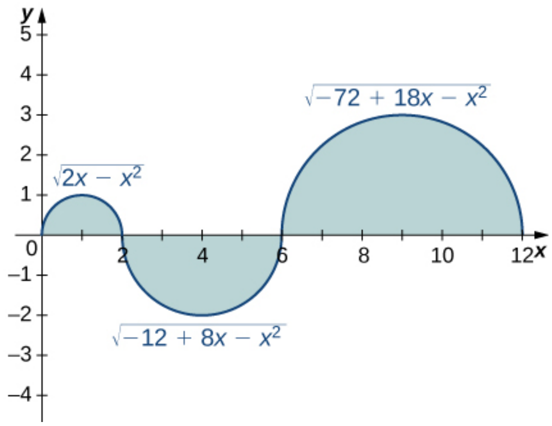

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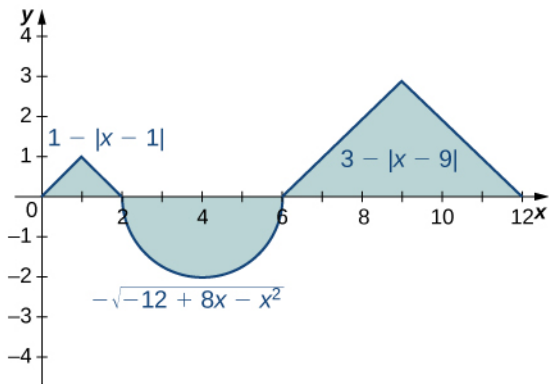


In the following exercises, evaluate the integral using area formulas.

76. $\int_0^3 (3 - x)dx$

77. $\int_2^3 (3 - x)dx$

78. $\int_{-3}^3 (3 - |x|)dx$

79. $\int_0^6 (3 - |x - 3|)dx$

80. $\int_{-2}^2 \sqrt{4 - x^2}dx$

81. $\int_1^5 \sqrt{4 - (x - 3)^2}dx$

82. $\int_0^{12} \sqrt{36 - (x - 6)^2}dx$

83. $\int_{-2}^3 (3 - |x|)dx$

In the following exercises, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

84. $\{(0, 0), (2, 1), (4, 3), (5, 0), (6, 0), (8, 3)\}$ over $[0, 8]$

85. $\{(0, 2), (1, 0), (3, 5), (5, 5), (6, 2), (8, 0)\}$ over $[0, 8]$

86. $\{(-4, -4), (-2, 0), (0, -2), (3, 3), (4, 3)\}$ over $[-4, 4]$

87. $\{(-4, 0), (-2, 2), (0, 0), (1, 2), (3, 2), (4, 0)\}$ over $[-4, 4]$

Suppose that $\int_0^4 f(x)dx = 5$ and $\int_0^2 f(x)dx = -3$, and $\int_0^4 g(x)dx = -1$ and $\int_0^2 g(x)dx = 2$. In the following exercises, compute the integrals.

88. $\int_0^4 (f(x) + g(x))dx$

89. $\int_2^4 (f(x) + g(x))dx$

90. $\int_0^2 (f(x) - g(x))dx$

91. $\int_2^4 (f(x) - g(x))dx$

92. $\int_0^2 (3f(x) - 4g(x))dx$

93. $\int_2^4 (4f(x) - 3g(x))dx$

In the following exercises, use the identity $\int_{-A}^A f(x)dx = \int_{-A}^0 f(x)dx + \int_0^A f(x)dx$ to compute the integrals.

94. $\int_{-\pi}^{\pi} \frac{\sin t}{1 + t^2}dt$ ($Hint$: $\sin(-t) = -\sin(t)$)

95. $\displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t}dt$

In the following exercises, find the net signed area between $f(x)$ and the x-axis.

96. $\displaystyle\int_1^3 (2-x)dx$ (*Hint:* Look at the graph of *f*.)

97. $\displaystyle\int_2^4 (x-3)^3\,dx$ (*Hint:* Look at the graph of *f*.)

In the following exercises, given that $\displaystyle\int_0^1 xdx = \frac{1}{2}$, $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$, and $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}$, compute the integrals.

98. $\displaystyle\int_0^1 \left(1+x+x^2+x^3\right)dx$

99. $\displaystyle\int_0^1 \left(1-x+x^2-x^3\right)dx$

100. $\displaystyle\int_0^1 (1-x)^2\,dx$

101. $\displaystyle\int_0^1 (1-2x)^3\,dx$

102. $\displaystyle\int_0^1 \left(6x-\frac{4}{3}x^2\right)dx$

103. $\displaystyle\int_0^1 \left(7-5x^3\right)dx$

In the following exercises, use the **comparison theorem**.

104. Show that $\displaystyle\int_0^3 \left(x^2-6x+9\right)dx \ge 0$.

105. Show that $\displaystyle\int_{-2}^3 (x-3)(x+2)dx \le 0$.

106. Show that $\displaystyle\int_0^1 \sqrt{1+x^3}dx \le \int_0^1 \sqrt{1+x^2}dx$.

107. Show that $\displaystyle\int_1^2 \sqrt{1+x}dx \le \int_1^2 \sqrt{1+x^2}dx$.

108. Show that $\displaystyle\int_0^{\pi/2} \sin t dt \ge \frac{\pi}{4}$. ($Hint$: $\sin t \ge \frac{2t}{\pi}$ over $\left[0, \frac{\pi}{2}\right]$)

109. Show that $\displaystyle\int_{-\pi/4}^{\pi/4} \cos t dt \ge \pi\sqrt{2}/4$.

In the following exercises, find the average value f_{ave} of f between a and b, and find a point c, where $f(c) = f_{\text{ave}}$.

110. $f(x) = x^2$, $a = -1$, $b = 1$

111. $f(x) = x^5$, $a = -1$, $b = 1$

112. $f(x) = \sqrt{4-x^2}$, $a = 0$, $b = 2$

113. $f(x) = (3-|x|)$, $a = -3$, $b = 3$

114. $f(x) = \sin x$, $a = 0$, $b = 2\pi$

115. $f(x) = \cos x$, $a = 0$, $b = 2\pi$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

116. **[T]** $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

117. **[T]** $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e}-1)$.

118. **[T]** $y = \tan x$ over the interval $\left[0, \frac{\pi}{4}\right]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

119. **[T]** $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1, 1]$; the exact solution is $\frac{\pi}{6}$.

In the following exercises, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

120. **[T]** $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

121. **[T]** $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}\left(e^4 - 1\right)$.

122. **[T]** $y = \left(\frac{1}{2}\right)^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64\ln(2)}$.

123. **[T]** $y = x\sin\left(x^2\right)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos\left(\pi^2\right) - 1}{2\pi}$.

124. Suppose that $A = \int_0^{2\pi} \sin^2 t dt$ and $B = \int_0^{2\pi} \cos^2 t dt$. Show that $A + B = 2\pi$ and $A = B$.

125. Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t dt = \pi$ and $B = \int_{-\pi/4}^{\pi/4} \tan^2 t dt$. Show that $A - B = \frac{\pi}{2}$.

126. Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to 1/2 Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to 1/2.

127. Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$.

128. Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)dt > \int_a^b \ell(t)dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t)dt$ whenever $a \le c < d \le b$.

129. Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward $(a < 0)$ and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B \left(ax^2 + bx + c\right)dx$ as large as possible?

130. Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \ge 0$ over $[a_{i-1}, a_i]$ or $f \le 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)dt$.

a. Explain why $\int_a^b f(t)dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$.

131. Suppose f and g are continuous functions such that $\int_c^d f(t)dt \le \int_c^d g(t)dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \le g(x)$ for all values of x.

132. Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

133. Suppose that $[a, b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1,\ldots, N$. Explain why the average value of f over $[a, b]$ is also equal to 1.

134. Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)dt = i$. Show that $\int_0^N f(t)dt = \frac{N(N+1)}{2}$.

135. Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)dt = i^2$. Show that $\int_0^N f(t)dt = \frac{N(N+1)(2N+1)}{6}$.

136. **[T]** Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10} + R_{10}}{2}$ for $f(t) = t^2$ over $[0, 1]$. Given that $\int_0^1 t^2 dt = 0.3\overline{3}$, to how many decimal places is $\frac{L_{10} + R_{10}}{2}$ accurate?

137. **[T]** Compute the left and right Riemann sums, L_{10} and R_{10}, and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = \left(4 - t^2\right)$ over $[1, 2]$. Given that $\int_1^2 \left(4 - t^2\right) dt = 1.\overline{66}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

138. If $\int_1^5 \sqrt{1 + t^4}\, dt = 41.7133...$, what is $\int_1^5 \sqrt{1 + u^4}\, du$?

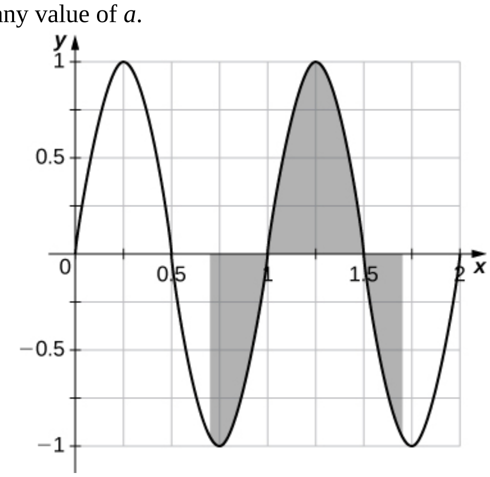
139. Estimate $\int_0^1 t dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t dt$?

140. Estimate $\int_0^1 t dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t dt$?

141. From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t) dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t) dt = 0$ for any value of a.



142. If f is 1-periodic $\left(f(t + 1) = f(t)\right)$, odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t) dt = 0$?

143. If f is 1-periodic and $\int_0^1 f(t) dt = A$, is it necessarily true that $\int_a^{1+a} f(t) dt = A$ for all A?