5.7 | Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

5.7.1 Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from **Functions and Graphs** that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in **Derivatives**, we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1.
$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{|a|} + C \tag{5.23}$$

2.
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{5.24}$$

3.
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{u}{|a|} + C \tag{5.25}$$

Proof

Let $y = \sin^{-1}\frac{x}{a}$. Then $a \sin y = x$. Now let's use implicit differentiation. We obtain

$$\begin{aligned} \frac{d}{dx}(a \sin y) &= \frac{d}{dx}(x) \\ a \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{a \cos y}. \end{aligned}$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1 - \sin^2 y}$. This gives

$$\begin{aligned} \frac{1}{a\cos y} &= \frac{1}{a\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - a^2\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Then for $-a \le x \le a,$ and generalizing to u, we have

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example 5.49

Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_0^{\frac{1}{2}} \\ &= \sin^{-1}\frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6}. \end{aligned}$$

5.40 Find the antiderivative of $\displaystyle\int \frac{dx}{\sqrt{1-16x^2}}$.

Example 5.50

Finding an Antiderivative Involving an Inverse Trigonometric Function

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$.

Solution

Substitute $u = 3x$. Then $du = 3dx$ and we have

$$\int \frac{dx}{\sqrt{4 - 9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4 - u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{4 - 9x^2}} &= \frac{1}{3}\int \frac{du}{\sqrt{4 - u^2}} \\ &= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C. \end{aligned}$$

5.41 Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9 - x^2}}$.

Example 5.51

Evaluating a Definite Integral

Evaluate the definite integral $\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1 - u^2}}$.

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\begin{aligned} \int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1 - u^2}} &= \sin^{-1} u \Big|_0^{\sqrt{3}/2} \\ &= \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\sin^{-1}(0)\right] \\ &= \frac{\pi}{3}. \end{aligned}$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 5.52

Finding an Antiderivative Involving the Inverse Tangent Function

Find an antiderivative of $\int \frac{1}{1+4x^2}dx.$

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C$. So we use substitution, letting $u = 2x$, then $du = 2dx$ and $1/2du = dx$. Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

5.42 Use substitution to find the antiderivative $\int \frac{dx}{25+4x^2}$.

Example 5.53

Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9+x^2}dx.$

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

5.43 Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Example 5.54

Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution

Use the formula for the inverse tangent. We have

$$\begin{aligned}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} \\ &= \left[\tan^{-1}\left(\sqrt{3}\right)\right] - \left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right] \\ &= \frac{\pi}{6}.\end{aligned}$$

5.44 Evaluate the definite integral $\displaystyle\int_0^2 \frac{dx}{4+x^2}$.