CHAPTER 5 REVIEW

KEY TERMS

average value of a function (or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

change of variables the substitution of a variable, such as u, for an expression in the integrand

definite integral a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

fundamental theorem of calculus the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1 uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2 (also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

integrable function a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand the function to the right of the integration symbol; the integrand includes the function being integrated

integration by substitution a technique for integration that allows integration of functions that are the result of a chain-rule derivative

left-endpoint approximation an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

limits of integration these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

lower sum a sum obtained by using the minimum value of $f(x)$ on each subinterval

mean value theorem for integrals guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

net change theorem if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

net signed area the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

partition a set of points that divides an interval into subintervals

regular partition a partition in which the subintervals all have the same width

riemann sum an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation (also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

total area total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

upper sum a sum obtained by using the maximum value of $f(x)$ on each subinterval

variable of integration indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

KEY EQUATIONS

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

- **Definite Integral**

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x) \text{ for constant } c$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- **Mean Value Theorem for Integrals**

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$

- **Fundamental Theorem of Calculus Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x).$

- **Fundamental Theorem of Calculus Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a).$

- **Net Change Theorem**

  $F(b) = F(a) + \int_a^b F'(x)dx$ or $\int_a^b F'(x)dx = F(b) - F(a)$

- **Substitution with Indefinite Integrals**

  $\int f[g(x)]g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$

- **Substitution with Definite Integrals**

  $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

- **Integrals of Exponential Functions**

  $\int e^x dx = e^x + C$

  $\int a^x dx = \frac{a^x}{\ln a} + C$

- **Integration Formulas Involving Logarithmic Functions**

  $\int x^{-1} dx = \ln|x| + C$

  $\int \ln x\, dx = x\ln x - x + C = x(\ln x - 1) + C$

  $\int \log_a x\, dx = \frac{x}{\ln a}(\ln x - 1) + C$

- **Integrals That Produce Inverse Trigonometric Functions**

  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

  $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

  $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

KEY CONCEPTS

5.1 Approximating Areas

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.

- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- The width of each rectangle is $\Delta x = \frac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of $\{x_i^*\}$ are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points $\{x_i^*\}$ at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

5.2 The Definite Integral

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

5.3 The Fundamental Theorem of Calculus

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function. See **The Mean Value Theorem for Integrals**.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral. See **Fundamental Theorem of Calculus, Part 1**.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula. See **The Fundamental Theorem of Calculus, Part 2**.

5.4 Integration Formulas and the Net Change Theorem

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

5.5 Substitution

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term ‘substitution’ refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

5.6 Integrals Involving Exponential and Logarithmic Functions

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

5.7 Integrals Resulting in Inverse Trigonometric Functions

- Formulas for derivatives of inverse trigonometric functions developed in **Derivatives of Exponential and Logarithmic Functions** lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

CHAPTER 5 REVIEW EXERCISES

True or False. Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

439. If $f(x) > 0,\ f'(x) > 0$ for all $x,$ then the right-hand rule underestimates the integral $\int_a^b f(x).$ Use a graph to justify your answer.

440. $\int_a^b f(x)^2\,dx = \int_a^b f(x)dx \int_a^b f(x)dx$

441. If $f(x) \le g(x)$ for all $x \in [a, b],$ then $\int_a^b f(x) \le \int_a^b g(x).$

442. All continuous functions have an antiderivative.

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

443. $y = 3x^2 - 2x + 1$ over $[-1, 1]$

444. $y = \ln\left(x^2 + 1\right)$ over $[0, e]$

445. $y = x^2 \sin x$ over $[0, \pi]$

446. $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Evaluate the following integrals.

447. $\int_{-1}^{1}\left(x^3 - 2x^2 + 4x\right)dx$

448. $\int_0^4 \frac{3t}{\sqrt{1 + 6t^2}}dt$

449. $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)d\theta$

450. $\int_0^{\pi/4} e^{\cos^2 x}\sin x \cos x\,dx$

Find the antiderivative.

451. $\int \frac{dx}{(x + 4)^3}$

452. $\int x\ln\left(x^2\right)dx$

453. $\int \frac{4x^2}{\sqrt{1 - x^6}}dx$

454. $\int \frac{e^{2x}}{1 + e^{4x}}dx$

Find the derivative.

455. $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1 + x^2}}dx$

456. $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

457. $\frac{d}{dx}\int_1^{\ln(x)} \left(4t+e^t\right)dt$

458. $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	−5,468,750
1990	−755,495
1995	−73,005
2000	−29,768
2005	−918
2010	−177

459. If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

460. The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,\ 500,\ 000(0.65)^t$, where t is measured in years since 1980, and C is cost in US$. Find the average cost per gigabyte of RAM for 1980 to 2010.

461. Find the average cost of 1GB RAM for 2005 to 2010.

462. The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686,$ where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \leq t \leq 0.5.$ What is the total distance the bullet travels in 0.5 sec?

463. What is the average velocity of the bullet for the first half-second?