4.9 | Newton's Method

Learning Objectives

4.9.1 Describe the steps of Newton's method.
4.9.2 Explain what an iterative process means.
4.9.3 Recognize when Newton's method does not work.
4.9.4 Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0.$ For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the zeroes of functions. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0.$ If f is the first-degree polynomial $f(x) = ax + b,$ then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}.$ If f is the second-degree polynomial $f(x) = ax^2 + bx + c,$ the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0.$ Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0.$ No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0.$ By sketching a graph of $f,$ we can estimate a root of $f(x) = 0.$ Let's call this estimate $x_0.$ We then draw the tangent line to f at $x_0.$ If $f'(x_0) \neq 0,$ this tangent line intersects the x-axis at some point $(x_1, 0).$ Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at $x_1.$ If $f'(x_1) \neq 0,$ this tangent line also intersects the x-axis, producing another approximation, $x_2.$ We continue in this way, deriving a list of approximations: $x_0, x_1, x_2, \ldots.$ Typically, the numbers $x_0, x_1, x_2, \ldots$ quickly approach an actual root $x^*,$ as shown in the following figure.

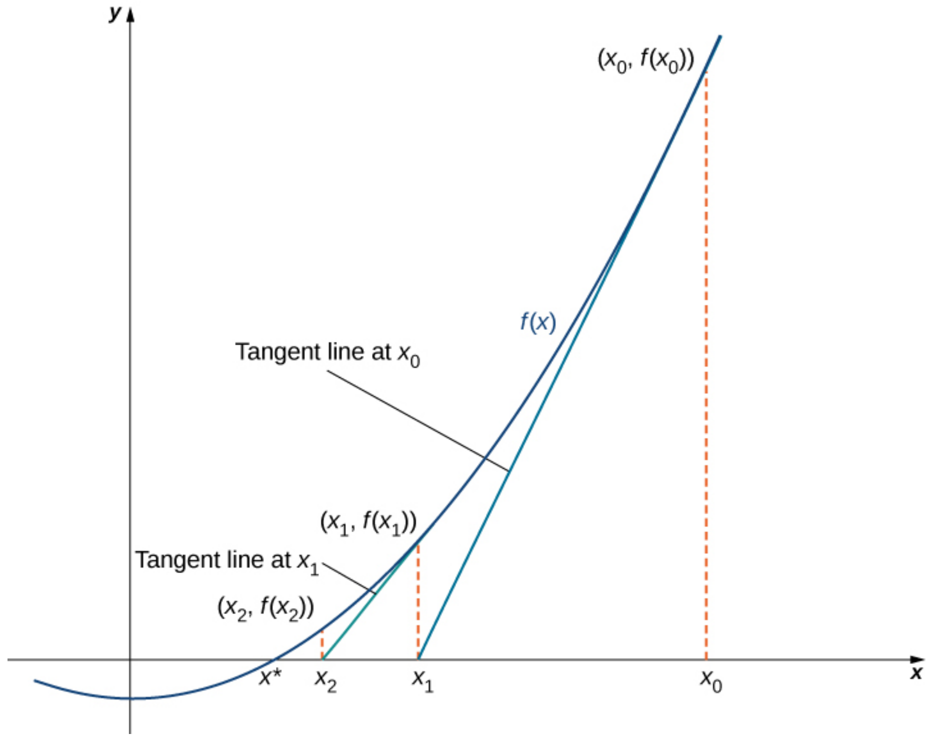

Figure 4.77 The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{4.8}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 4.46

Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1, x_2, x_3, x_4,$ and x_5.

Solution

From **Figure 4.78**, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use **Equation 4.8**. Since $f(x) = x^3 - 3x + 1,$ the derivative is $f'(x) = 3x^2 - 3$. Using **Equation 4.8** with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, $x_2,$ we use **Equation 4.8** with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 = \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

$$\begin{aligned} x_1 &\approx 1.666666667 \\ x_2 &\approx 1.548611111 \\ x_3 &\approx 1.532390162 \\ x_4 &\approx 1.532088989 \\ x_5 &\approx 1.532088886 \\ x_6 &\approx 1.532088886. \end{aligned}$$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

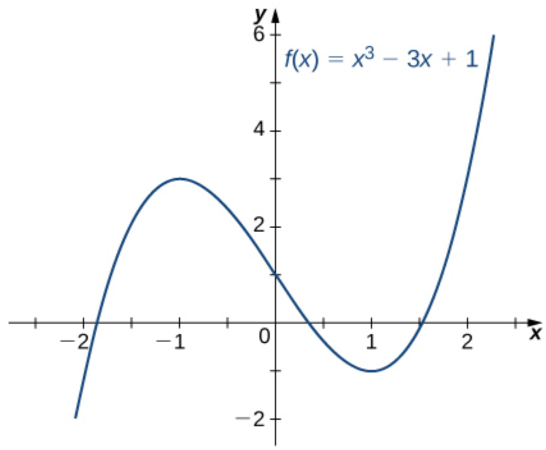

Figure 4.78 The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

4.45 Letting $x_0 = 0,$ let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 4.47

Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (**Figure 4.79**). Let $f(x) = x^2 - 2,$ let $x_0 = 2,$ and calculate x_1, x_2, x_3, x_4, x_5. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2},$ the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$.)

Solution

For $f(x) = x^2 - 2,\ f'(x) = 2x$. From **Equation 4.8**, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x^2_{n-1} - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5 \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667. \end{aligned}$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562. \end{aligned}$$

Since we obtained the same value for x_4 and $x_5,$ it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.

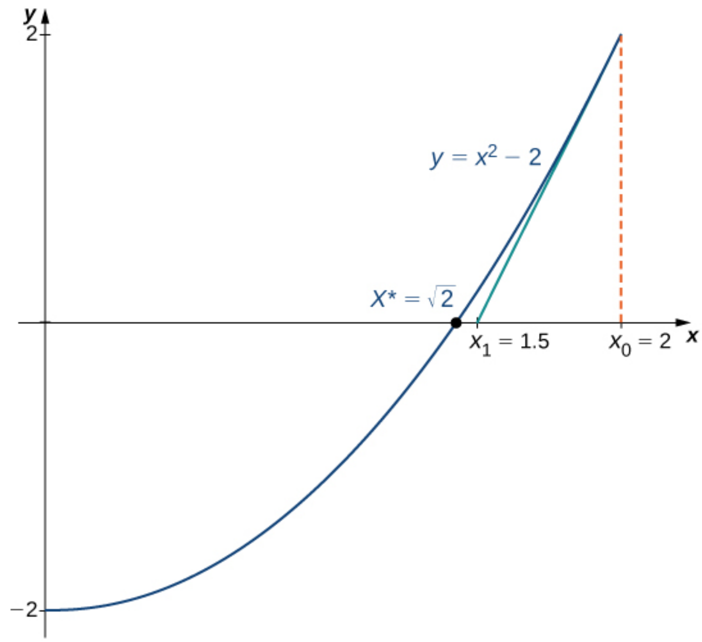

Figure 4.79 We can use Newton's method to find $\sqrt{2}$.

4.46 Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3.$ Find x_1 and x_2.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right],$ we can rewrite **Equation 4.8** as $x_n = F(x_{n-1}).$ This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations $x_n,$ the derivative f' is zero at $x_n,$ but $f(x_n) \neq 0.$ As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see **Figure 4.80**). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In **Example 4.48**, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

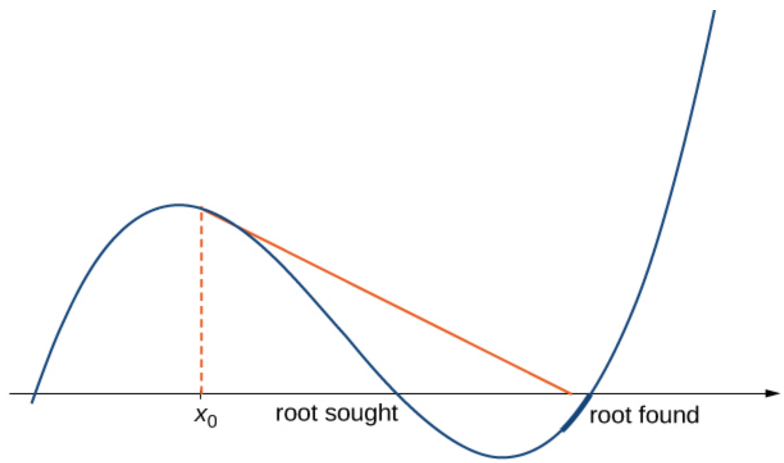

Figure 4.80 If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.48

When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (see **Figure 4.81**). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

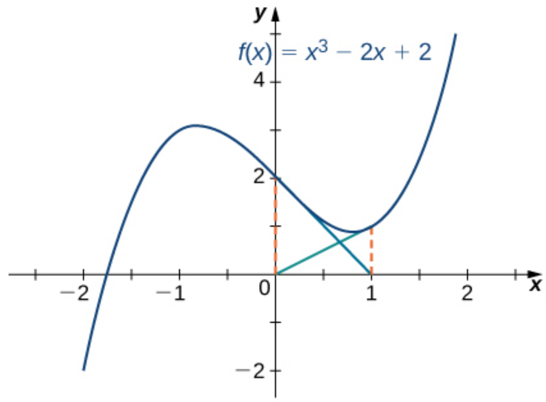

Figure 4.81 The approximations continue to alternate between 0 and 1 and never approach the root of f.

4.47 For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

From **Example 4.48**, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In **Example 4.49**, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 4.49

Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3 \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

$$x_1 = \frac{1}{2}(0) + 4 = 4$$
$$x_2 = \frac{1}{2}(4) + 4 = 6$$
$$x_3 = \frac{1}{2}(6) + 4 = 7$$
$$x_4 = \frac{1}{2}(7) + 4 = 7.5$$
$$x_5 = \frac{1}{2}(7.5) + 4 = 7.75$$
$$x_6 = \frac{1}{2}(7.75) + 4 = 7.875$$
$$x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$$
$$x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$$
$$x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$$

From this list, we conjecture that the values x_n approach $8.$

Figure 4.82 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $\left(x_0, F(x_0)\right)$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $\left(x_1, F(x_1)\right)$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty$.

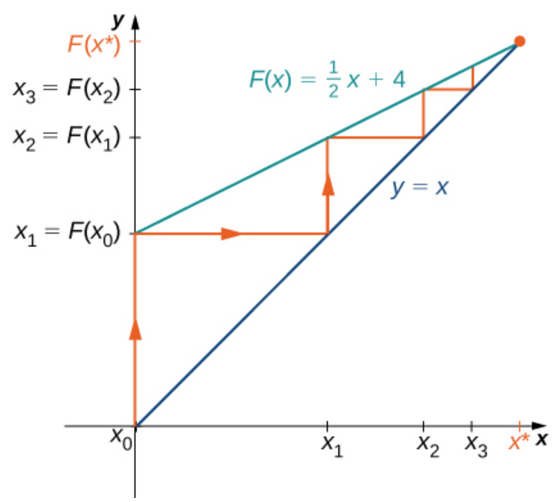

Figure 4.82 This iterative process approaches the value $x^* = 8.$

4.48 Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Student PROJECT

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in **Example 4.48** that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-*cycle*. Iterative processes can converge to cycles with various periodicities, such as $2-\text{cycles},\ 4-\text{cycles}$ (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call *chaos*. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the Mandelbrot set (see **Figure 4.83**), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

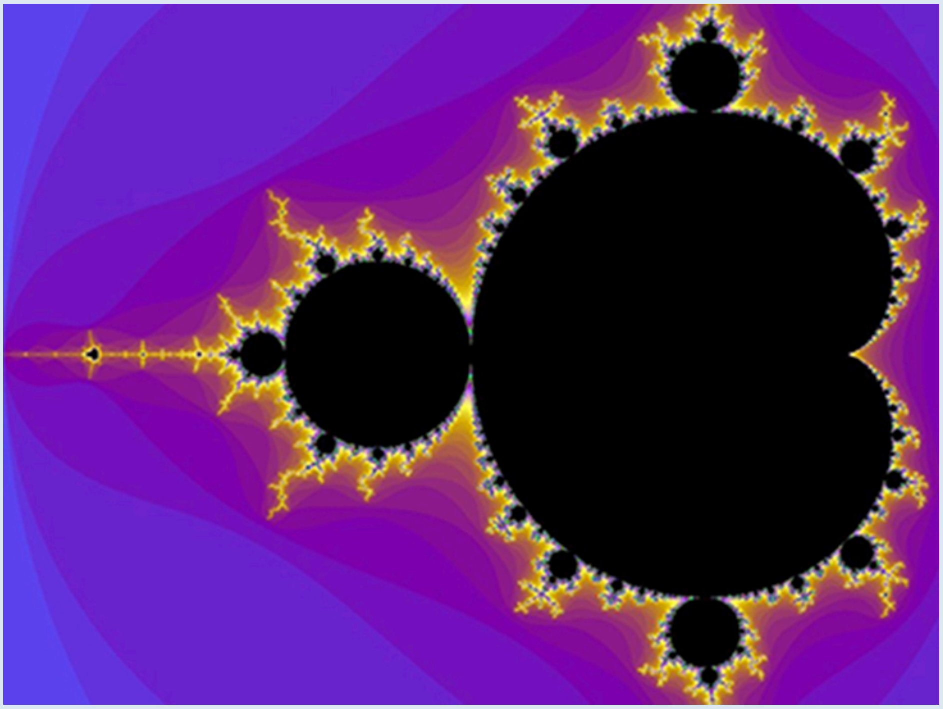

Figure 4.83 The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1-x), \text{ where } x \in [0,\ 1] \text{ and } r > 0$$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a *cobweb diagram*. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. **Figure 4.84** shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

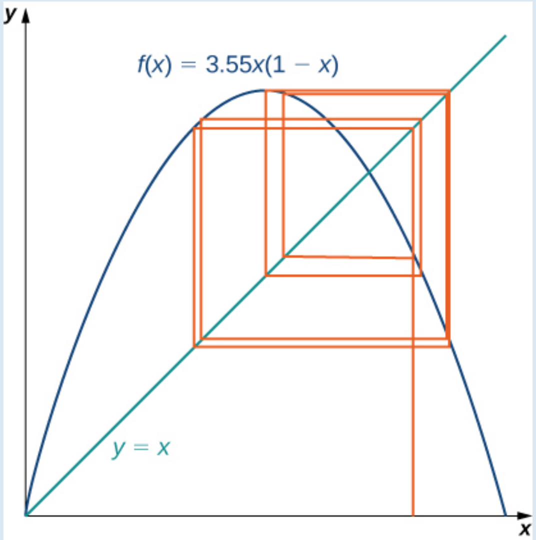

Figure 4.84 A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, $2 - \text{cycle}, 4 - \text{cycle}.$)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?