CHAPTER 4 REVIEW

KEY TERMS

absolute extremum if f has an absolute maximum or absolute minimum at $c,$ we say f has an absolute extremum at c

absolute maximum if $f(c) \geq f(x)$ for all x in the domain of $f,$ we say f has an absolute maximum at c

absolute minimum if $f(c) \leq f(x)$ for all x in the domain of $f,$ we say f has an absolute minimum at c

antiderivative a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

concave down if f is differentiable over an interval I and f' is decreasing over $I,$ then f is concave down over I

concave up if f is differentiable over an interval I and f' is increasing over $I,$ then f is concave up over I

concavity the upward or downward curve of the graph of a function

concavity test suppose f is twice differentiable over an interval I; if $f'' > 0$ over $I,$ then f is concave up over I; if $f'' < 0$ over $I,$ then f is concave down over I

critical point if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

differential the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)dx$

differential form given a differentiable function $y = f'(x),$ the equation $dy = f'(x)dx$ is the differential form of the derivative of y with respect to x

end behavior the behavior of a function as $x \to \infty$ and $x \to -\infty$

extreme value theorem if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem if f has a local extremum at $c,$ then c is a critical point of f

first derivative test let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through $c,$ then f has a local maximum at c; if f' changes sign from negative to positive as x increases through $c,$ then f has a local minimum at c; if f' does not change sign as x increases through $c,$ then f does not have a local extremum at c

horizontal asymptote if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L,$ then $y = L$ is a horizontal asymptote of f

indefinite integral the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)dx$ to denote the indefinite integral of f

indeterminate forms when evaluating a limit, the forms $\frac{0}{0},$ $\infty/\infty,$ $0 \cdot \infty,$ $\infty - \infty,$ $0^0,$ $\infty^0,$ and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is

infinite limit at infinity a function that becomes arbitrarily large as x becomes large

inflection point if f is continuous at c and f changes concavity at $c,$ the point $(c, f(c))$ is an inflection point of f

initial value problem a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

iterative process process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

limit at infinity the limiting value, if it exists, of a function as $x \to \infty$ or $x \to -\infty$

linear approximation the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

local extremum if f has a local maximum or local minimum at $c,$ we say f has a local extremum at c

local maximum if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I,$ we say f has a local maximum at c

local minimum if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I,$ we say f has a local minimum at c

L'Hôpital's rule if f and g are differentiable functions over an interval $a,$ except possibly at $a,$ and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$ assuming the limit on the right exists or is ∞ or $-\infty$

mean value theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Newton's method method for approximating roots of $f(x) = 0;$ using an initial guess $x_0;$ each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

oblique asymptote the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or $x \to -\infty$

optimization problems problems that are solved by finding the maximum or minimum value of a function

percentage error the relative error expressed as a percentage

propagated error the error that results in a calculated quantity $f(x)$ resulting from a measurement error *dx*

related rates are rates of change associated with two or more related quantities that are changing over time

relative error given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

rolle's theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ and if $f(a) = f(b),$ then there exists $c \in (a, b)$ such that $f'(c) = 0$

second derivative test suppose $f'(c) = 0$ and f'' is continuous over an interval containing $c;$ if $f''(c) > 0,$ then f has a local minimum at $c;$ if $f''(c) < 0,$ then f has a local maximum at $c;$ if $f''(c) = 0,$ then the test is inconclusive

tangent line approximation (linearization) since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

KEY EQUATIONS

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**
  $dy = f'(x)dx.$

KEY CONCEPTS

4.1 Related Rates

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

4.2 Linear Approximations and Differentials

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

$$dy = f'(x)dx$$

  is an approximation for the change in y. The actual change in y is

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the *propagated error*. The propagated error can be estimated by

$$dy \approx f'(x)dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

4.3 Maxima and Minima

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

4.4 The Mean Value Theorem

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = 0 = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is the Mean Value Theorem.

- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

4.5 Derivatives and the Shape of a Graph

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.
- If $f''(x) > 0$ over an interval I, then f is concave up over I.
- If $f''(x) < 0$ over an interval I, then f is concave down over I.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

4.6 Limits at Infinity and Asymptotes

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0$, $p(x)$ approaches ∞ or $-\infty$ at each end.
- For a rational function $f(x) = \frac{p(x)}{q(x)}$, the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

4.7 Applied Optimization Problems

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.

- Look for critical points to locate local extrema.

4.8 L'Hôpital's Rule

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function x^p, $p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function x^p, $p > 0$.

4.9 Newton's Method

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1, x_2, x_3, \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0, x_1, x_2, \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0, x_1, x_2, \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

4.10 Antiderivatives

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \; y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

CHAPTER 4 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

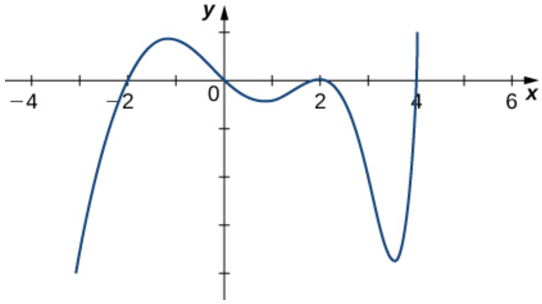
525. If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

526. If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

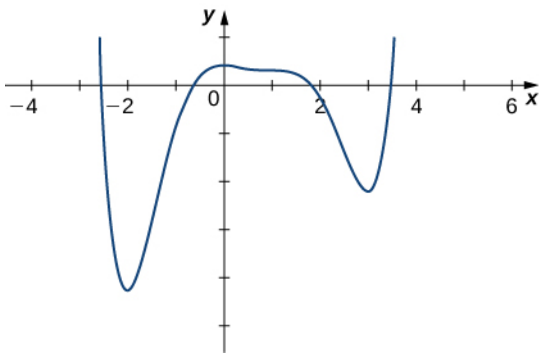
527. There is a function such that $f(x) < 0$, $f'(x) > 0$, and $f''(x) < 0$. (A graphical "proof" is acceptable for this answer.)

528. There is a function such that there is both an inflection point and a critical point for some value $x = a$.

529. Given the graph of f', determine where f is increasing or decreasing.



530. The graph of f is given below. Draw f'.



531. Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

532. Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

533. $f(x) = x + \sin^2(x)$ over $[0, \pi]$

534. $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

535. $x(t) = 3t^4 - 8t^3 - 18t^2$

536. $y = x + \sin(\pi x)$

537. $g(x) = x - \sqrt{x}$

538. $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

539. $\lim_{x \to \infty} \frac{3x\sqrt{x^2 + 1}}{\sqrt{x^4 - 1}}$

540. $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right)$

541. $\lim_{x \to 1} \frac{x - 1}{\sin(\pi x)}$

542. $\lim_{x \to \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

543. $y = x^3 + 1,\ x_0 = 0.5$

544. $\frac{1}{x + 1} = \frac{1}{2},\ x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

545. $g(x) = \sqrt{x} - \frac{1}{x^2}$

546. $f(x) = 2x + 6\cos x,\ F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

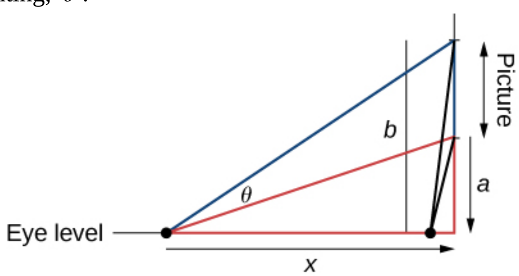
547. $y = \frac{1}{x(x + 1)^2}$

548. $y = x - \sqrt{4 - x^2}$

549. A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2\ \text{m}^3$/sec. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

550. A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right)m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

551. The famous Regiomontanus' problem for angle maximization was proposed during the 15 th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



552. An airline sells tickets from Tokyo to Detroit for $\$1200$. There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?