

2.3 | The Limit Laws

Learning Objectives

- 2.3.1** Recognize the basic limit laws.
- 2.3.2** Use the limit laws to evaluate the limit of a function.
- 2.3.3** Evaluate the limit of a function by factoring.
- 2.3.4** Use the limit laws to evaluate the limit of a polynomial or rational function.
- 2.3.5** Evaluate the limit of a function by factoring or by using conjugates.
- 2.3.6** Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in **Two Important Limits** and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.4: Basic Limit Results

For any real number a and any constant c ,

$$\text{i. } \lim_{x \rightarrow a} x = a \quad (2.14)$$

$$\text{ii. } \lim_{x \rightarrow a} c = c \quad (2.15)$$

Example 2.13

Evaluating a Basic Limit

Evaluate each of the following limits using **Basic Limit Results**.

a. $\lim_{x \rightarrow 2} x$

b. $\lim_{x \rightarrow 2} 5$

Solution

a. The limit of x as x approaches a is a : $\lim_{x \rightarrow 2} x = 2$.

b. The limit of a constant is that constant: $\lim_{x \rightarrow 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a . Assume that L and M are real numbers such that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$

Difference law for limits: $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \rightarrow a} cf(x) = c \cdot \lim_{x \rightarrow a} f(x) = cL$

Product law for limits: $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = L^n$ for every positive integer n .

Root law for limits: $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

We now practice applying these limit laws to evaluate a limit.

Example 2.14**Evaluating a Limit Using Limit Laws**

Use the limit laws to evaluate $\lim_{x \rightarrow -3} (4x + 2)$.

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \rightarrow -3} (4x + 2) &= \lim_{x \rightarrow -3} 4x + \lim_{x \rightarrow -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \rightarrow -3} x + \lim_{x \rightarrow -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example 2.15**Using Limit Laws Repeatedly**

Use the limit laws to evaluate $\lim_{x \rightarrow 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{2x^2 - 3x + 1}{x^3 + 4} &= \frac{\lim_{x \rightarrow 2} (2x^2 - 3x + 1)}{\lim_{x \rightarrow 2} (x^3 + 4)} && \text{Apply the quotient law, making sure that } (2)^3 + 4 \neq 0 \\
 &= \frac{2 \cdot \lim_{x \rightarrow 2} x^2 - 3 \cdot \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x^3 + \lim_{x \rightarrow 2} 4} && \text{Apply the sum law and constant multiple law.} \\
 &= \frac{2 \cdot \left(\lim_{x \rightarrow 2} x \right)^2 - 3 \cdot \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1}{\left(\lim_{x \rightarrow 2} x \right)^3 + \lim_{x \rightarrow 2} 4} && \text{Apply the power law.} \\
 &= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
 \end{aligned}$$



2.11 Use the limit laws to evaluate $\lim_{x \rightarrow 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \rightarrow a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 2.6: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\begin{aligned}
 \lim_{x \rightarrow a} p(x) &= p(a) \\
 \lim_{x \rightarrow a} \frac{p(x)}{q(x)} &= \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.
 \end{aligned}$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}
 \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0) \\
 &= c_n \left(\lim_{x \rightarrow a} x \right)^n + c_{n-1} \left(\lim_{x \rightarrow a} x \right)^{n-1} + \cdots + c_1 \left(\lim_{x \rightarrow a} x \right) + \lim_{x \rightarrow a} c_0 \\
 &= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\
 &= p(a).
 \end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.16 applies this result.

Example 2.16

Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$



2.12 Evaluate $\lim_{x \rightarrow -2} (3x^3 - 2x + 7)$.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \rightarrow a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a , then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$.

To understand this idea better, consider the limit $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

The function

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in **Figure 2.24**.

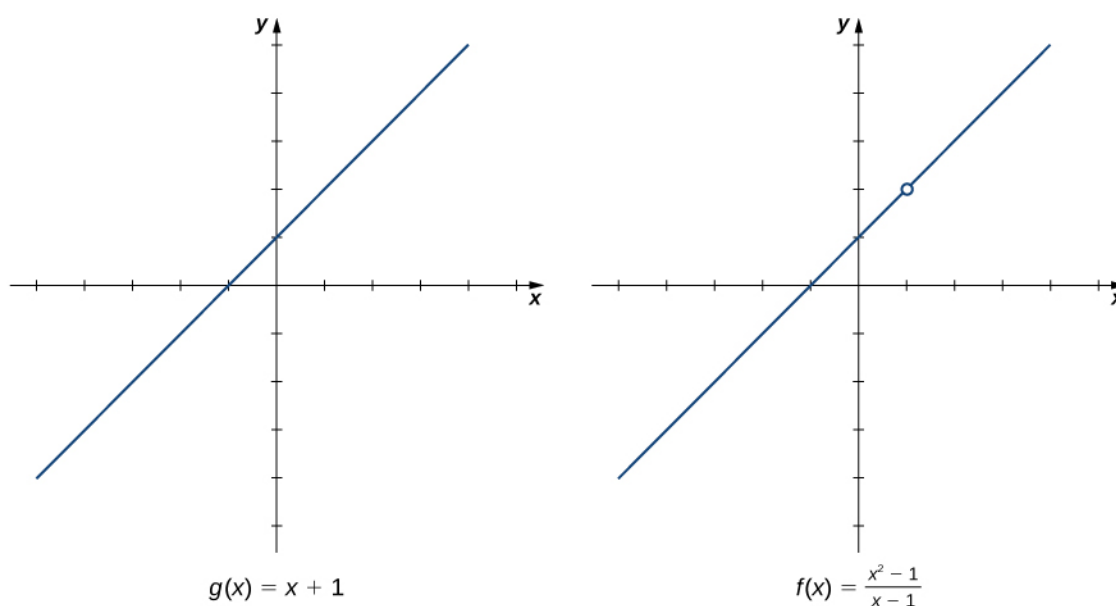


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 2. \end{aligned}$$

The limit has the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, where $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a . To do this, we may need to try one or more of the following steps:
 - a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
 - b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
 - c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. **Example 2.17** illustrates the factor-and-cancel technique; **Example 2.18** shows multiplying by a conjugate. In **Example 2.19**, we look at simplifying a complex fraction.

Example 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution

Step 1. The function $f(x) = \frac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{x(x - 3)}{(x - 3)(2x + 1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2 - 3x}{2x^2 - 5x - 3} = \frac{x}{2x + 1}$. Therefore,

$$\lim_{x \rightarrow 3} \frac{x(x - 3)}{(x - 3)(2x + 1)} = \lim_{x \rightarrow 3} \frac{x}{2x + 1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x \rightarrow 3} \frac{x}{2x + 1} = \frac{3}{7}.$$



2.13

Evaluate $\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 - 9}$.

Example 2.18

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x \rightarrow -1} \frac{\sqrt{x+2} - 1}{x + 1}$.

Solution

Step 1. $\frac{\sqrt{x+2} - 1}{x + 1}$ has the form $0/0$ at -1 . Let's begin by multiplying by $\sqrt{x+2} + 1$, the conjugate of $\sqrt{x+2} - 1$, on the numerator and denominator:

$$\lim_{x \rightarrow -1} \frac{\sqrt{x+2} - 1}{x + 1} = \lim_{x \rightarrow -1} \frac{\sqrt{x+2} - 1}{x + 1} \cdot \frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x + 1)$ in the denominator cancels out in the end:

$$= \lim_{x \rightarrow -1} \frac{x + 1}{(x + 1)(\sqrt{x+2} + 1)}.$$

Step 3. Then we cancel:

$$= \lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$



2.14 Evaluate $\lim_{x \rightarrow 5} \frac{\sqrt{x-1}-2}{x-5}$.

Example 2.19

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution

Step 1. $\frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$= \lim_{x \rightarrow 1} \frac{2 - (x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \rightarrow 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \rightarrow 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \rightarrow 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \rightarrow 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$



2.15

Evaluate $\lim_{x \rightarrow -3} \frac{\frac{1}{x+2} + 1}{x+3}$.

Example 2.20 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.20

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right)$.

Solution

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\begin{aligned} \frac{1}{x} + \frac{5}{x(x-5)} &= \frac{x-5+5}{x(x-5)} \\ &= \frac{x}{x(x-5)}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) &= \lim_{x \rightarrow 0} \frac{x}{x(x-5)} \\ &= \lim_{x \rightarrow 0} \frac{1}{x-5} \\ &= -\frac{1}{5}. \end{aligned}$$



2.16

Evaluate $\lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{4}{x^2 - 2x - 3} \right)$.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x \rightarrow a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a) ; for a limit of the form $\lim_{x \rightarrow a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c) . **Example 2.21** illustrates this point.

Example 2.21

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x \rightarrow 3^-} \sqrt{x-3}$

b. $\lim_{x \rightarrow 3^+} \sqrt{x-3}$

Solution

Figure 2.25 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

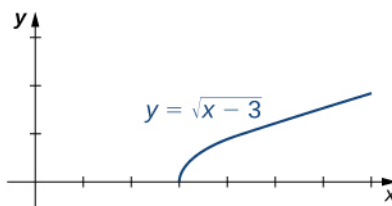


Figure 2.25 The graph shows the function $f(x) = \sqrt{x-3}$.

- a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \rightarrow 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \rightarrow 3^-} \sqrt{x-3}$ does not exist.
- b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \rightarrow 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \rightarrow 3^+} \sqrt{x-3} = 0$.

In **Example 2.22** we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

- a. $\lim_{x \rightarrow 2^-} f(x)$
- b. $\lim_{x \rightarrow 2^+} f(x)$
- c. $\lim_{x \rightarrow 2} f(x)$

Solution

Figure 2.26 illustrates the function $f(x)$ and aids in our understanding of these limits.

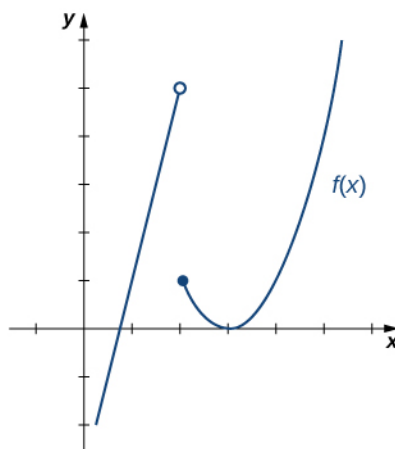


Figure 2.26 This graph shows a function $f(x)$.

- a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (4x - 3) = 5.$$

- b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x - 3)^2 = 1.$$

- c. Since $\lim_{x \rightarrow 2^-} f(x) = 5$ and $\lim_{x \rightarrow 2^+} f(x) = 1$, we conclude that $\lim_{x \rightarrow 2} f(x)$ does not exist.



2.17

Graph $f(x) = \begin{cases} -x - 2 & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ x^3 & \text{if } x > -1 \end{cases}$ and evaluate $\lim_{x \rightarrow -1^-} f(x)$.

We now turn our attention to evaluating a limit of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, where $\lim_{x \rightarrow a} f(x) = K$, where $K \neq 0$ and $\lim_{x \rightarrow a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0$, $K \neq 0$ at a .

Example 2.23

Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \rightarrow 2^-} \frac{x - 3}{x^2 - 2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the

left, the numerator approaches -1 ; and the denominator approaches 0 . Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-2x} = \lim_{x \rightarrow 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \rightarrow 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \rightarrow 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-2x} = +\infty.$$



2.18 Evaluate $\lim_{x \rightarrow 1} \frac{x+2}{(x-1)^2}$.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by “squeezing” a function, with a limit at a point a that is unknown, between two functions having a common known limit at a . **Figure 2.27** illustrates this idea.

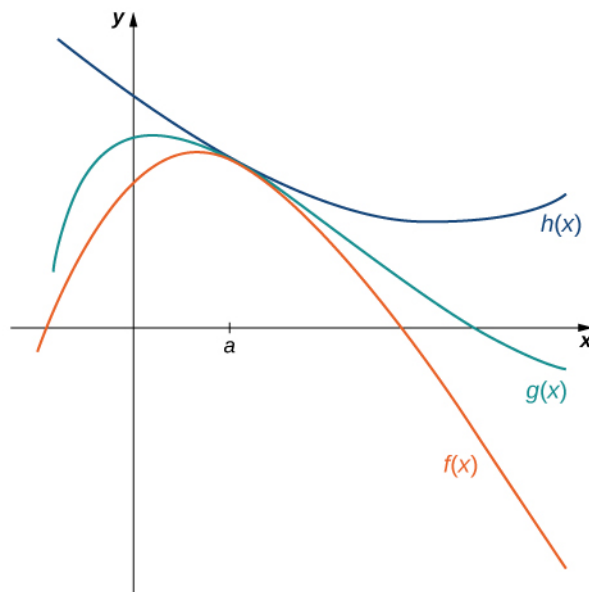


Figure 2.27 The Squeeze Theorem applies when $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$.

Theorem 2.7: The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a . If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where L is a real number, then $\lim_{x \rightarrow a} g(x) = L$.

Example 2.24**Applying the Squeeze Theorem**

Apply the squeeze theorem to evaluate $\lim_{x \rightarrow 0} x \cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x , we have $-|x| \leq x \cos x \leq |x|$. Since $\lim_{x \rightarrow 0} (-|x|) = 0 = \lim_{x \rightarrow 0} |x|$, from the squeeze theorem, we obtain $\lim_{x \rightarrow 0} x \cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x \cos x$, and $h(x) = |x|$ are shown in **Figure 2.28**.

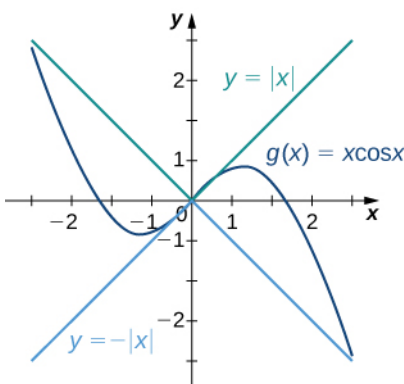


Figure 2.28 The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.



2.19 Use the squeeze theorem to evaluate $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \rightarrow 0} \sin \theta$. Consider the unit circle shown in **Figure 2.29**. In the figure, we see that $\sin \theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin \theta < \theta$.

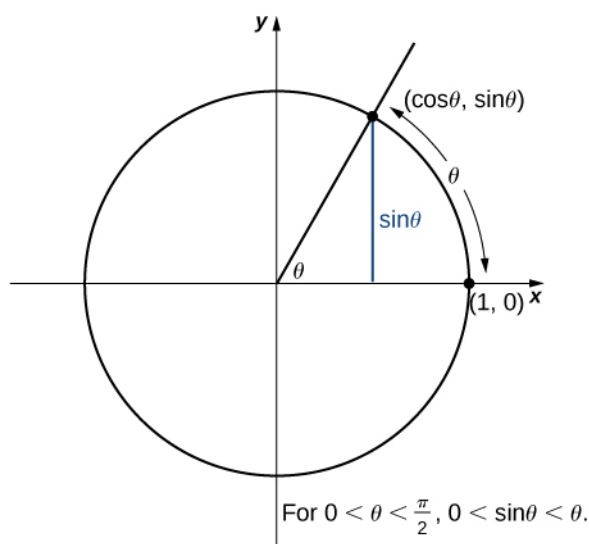


Figure 2.29 The sine function is shown as a line on the unit circle.

Because $\lim_{\theta \rightarrow 0^+} 0 = 0$ and $\lim_{\theta \rightarrow 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0.$$

To see that $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0$, $0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin \theta < -\theta$. It follows that $0 > \sin \theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta \rightarrow 0^+} \sin \theta = 0$ and $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$,

$$\lim_{\theta \rightarrow 0} \sin \theta = 0. \quad (2.16)$$

Next, using the identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta \rightarrow 0} \cos \theta = \lim_{\theta \rightarrow 0} \sqrt{1 - \sin^2 \theta} = 1. \quad (2.17)$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$. To evaluate this limit, we use the unit circle in **Figure 2.30**. Notice that this figure adds one additional triangle to **Figure 2.30**. We see that the length of the side opposite angle θ in this new triangle is $\tan \theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, $\sin \theta < \theta < \tan \theta$.

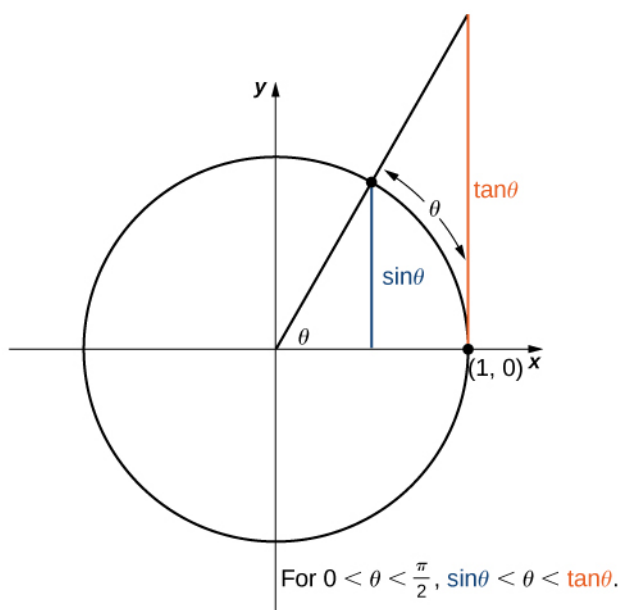


Figure 2.30 The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin \theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Equivalently, we have

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} 1 = 1 = \lim_{\theta \rightarrow 0^+} \cos \theta$, we conclude that $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$. By applying a manipulation similar to that used in demonstrating that $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$, we can show that $\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1$. Thus,

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1. \quad (2.18)$$

In **Example 2.25** we use this limit to establish $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 2.25

Evaluating an Important Trigonometric Limit

Evaluate $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\&= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\&= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} \\&= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \\&= 1 \cdot \frac{0}{2} = 0.\end{aligned}$$

Therefore,

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0. \quad (2.19)$$



2.20 Evaluate $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta}$.

Student PROJECT

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287–212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in **Figure 2.31** in terms of θ and r .

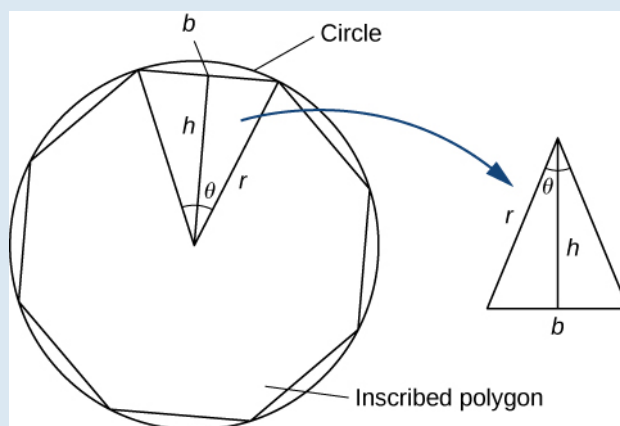


Figure 2.31

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r . (Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an n -sided regular polygon is inscribed in a circle of radius r , find a relationship between θ and n . Solve this for n . Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the n -sided polygon in terms of r and θ .
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (Hint: $\lim_{\theta \rightarrow 0} \frac{(\sin\theta)}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in **Introduction to Integration**.