CHAPTER 2 REVIEW

KEY TERMS

average velocity the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$

constant multiple law for limits the limit law $\lim\limits_{x \to a} cf(x) = c \cdot \lim\limits_{x \to a} f(x) = cL$

continuity at a point A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and (3) $\lim\limits_{x \to a} f(x) = f(a)$

continuity from the left A function is continuous from the left at b if $\lim\limits_{x \to b^-} f(x) = f(b)$

continuity from the right A function is continuous from the right at a if $\lim\limits_{x \to a^+} f(x) = f(a)$

continuity over an interval a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

difference law for limits the limit law $\lim\limits_{x \to a}(f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$

differential calculus the field of calculus concerned with the study of derivatives and their applications

discontinuity at a point A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

epsilon-delta definition of the limit $\lim\limits_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

infinite discontinuity An infinite discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$

infinite limit A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

instantaneous velocity The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus the study of integrals and their applications

Intermediate Value Theorem Let f be continuous over a closed bounded interval $[a, b]$; if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

intuitive definition of the limit If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

jump discontinuity A jump discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

limit the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

limit laws the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

multivariable calculus the study of the calculus of functions of two or more variables

one-sided limit A one-sided limit of a function is a limit taken from either the left or the right

power law for limits the limit law $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n

product law for limits the limit law $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

quotient law for limits the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

removable discontinuity A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists

root law for limits the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

secant A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$

squeeze theorem states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

tangent A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

triangle inequality If a and b are any real numbers, then $|a + b| \leq |a| + |b|$

vertical asymptote A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

KEY EQUATIONS

- **Slope of a Secant Line**
  $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$
- **Average Velocity over Interval** $[a, t]$
  $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$
- **Intuitive Definition of the Limit**
  $\lim_{x \to a} f(x) = L$
- **Two Important Limits**
  $\lim_{x \to a} x = a \quad \lim_{x \to a} c = c$
- **One-Sided Limits**

$\lim_{x \to a^-} f(x) = L \quad \lim_{x \to a^+} f(x) = L$

- **Infinite Limits from the Left**
  $\lim_{x \to a^-} f(x) = +\infty \quad \lim_{x \to a^-} f(x) = -\infty$
- **Infinite Limits from the Right**
  $\lim_{x \to a^+} f(x) = +\infty \quad \lim_{x \to a^+} f(x) = -\infty$
- **Two-Sided Infinite Limits**
  $\lim_{x \to a} f(x) = +\infty : \lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$
  $\lim_{x \to a} f(x) = -\infty : \lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$
- **Basic Limit Results**
  $\lim_{x \to a} x = a \ \lim_{x \to a} c = c$
- **Important Limits**
  $\lim_{\theta \to 0} \sin\theta = 0$
  $\lim_{\theta \to 0} \cos\theta = 1$
  $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
  $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$

KEY CONCEPTS

2.1 A Preview of Calculus

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the *x*-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

2.2 The Limit of a Function

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

2.3 The Limit Laws

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions, $\lim_{x \to a} f(x) = f(a)$.

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

2.4 Continuity

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L,$ then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

2.5 The Precise Definition of a Limit

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the *epsilon-delta definition of the limit*.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.

CHAPTER 2 REVIEW EXERCISES

True or False. In the following exercises, justify your answer with a proof or a counterexample.

208. A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

209. You can use the quotient rule to evaluate $\lim_{x \to 0} \frac{\sin x}{x}.$

210. If there is a vertical asymptote at $x = a$ for the function $f(x),$ then *f* is undefined at the point $x = a.$

211. If $\lim_{x \to a} f(x)$ does not exist, then *f* is undefined at the point $x = a.$

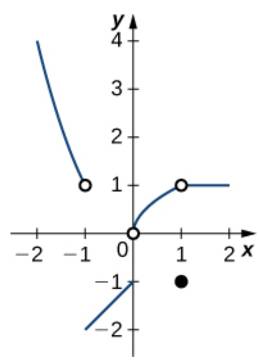
212. Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In the following exercises, evaluate the limit algebraically or explain why the limit does not exist.

213. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$

214. $\lim\limits_{x \to 0} 3x^2 - 2x + 4$

215. $\lim\limits_{x \to 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

216. $\lim\limits_{x \to \pi/2} \frac{\cot x}{\cos x}$

217. $\lim\limits_{x \to -5} \frac{x^2 + 25}{x + 5}$

218. $\lim\limits_{x \to 2} \frac{3x^2 - 2x - 8}{x^2 - 4}$

219. $\lim\limits_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

220. $\lim\limits_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$

221. $\lim\limits_{x \to 4} \frac{4 - x}{\sqrt{x} - 2}$

222. $\lim\limits_{x \to 4} \frac{1}{\sqrt{x} - 2}$

In the following exercises, use the squeeze theorem to prove the limit.

223. $\lim\limits_{x \to 0} x^2 \cos(2\pi x) = 0$

224. $\lim\limits_{x \to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$

225. Determine the domain such that the function $f(x) = \sqrt{x - 2} + xe^x$ is continuous over its domain.

In the following exercises, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

226. $f(x) = \begin{cases} x^2 + 1, & x > c \\ 2x, & x \le c \end{cases}$

227. $f(x) = \begin{cases} \sqrt{x + 1}, & x > -1 \\ x^2 + c, & x \le -1 \end{cases}$

In the following exercises, use the precise definition of limit to prove the limit.

228. $\lim\limits_{x \to 1} (8x + 16) = 24$

229. $\lim\limits_{x \to 0} x^3 = 0$

230. A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

231. A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

232. From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.